Penneys: Temperley Lieb

October-18-13
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- $\otimes$-category
- 2-category w/ 1-object:

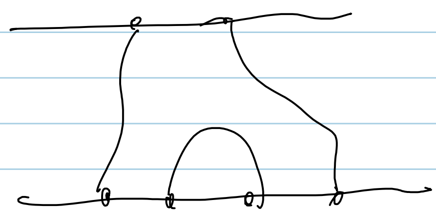
  0-morphisms $\{*\}$

  1-morphisms:

  $$\{1<2<3\ldots<n\} = [n] \quad n\geq 0$$

  $$[m]\circ[n] = [m+n]$$

2-morphisms: $\mathbb{C}$ or $\mathbb{C}(q)$-linear combinations of string diagrams



$$\begin{array}{c}[2]\\ \uparrow \\ [4]\end{array}$$

non-x'ing strings.

2 types of composition

horizontal join
vertical join [with circle-removal using $\bigcirc = [2]_q = q+q^{-1}$

$$TL(m,n) = 2\text{-mor}([m],[n])$$

$TL_n := TL(n,n)$ is an algebra

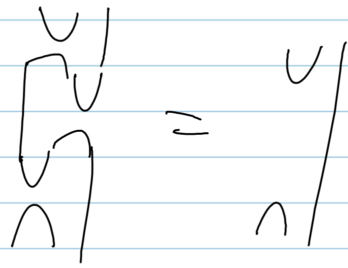
Generators & relations:

$$TL_n = \mathbb{C}\langle 1, E_1, \ldots E_{n-1} \mid \text{rels} \rangle$$

rels: 1. $E_i = [2]_q^{-1} E_i^2$

2. $E_i E_j = E_j E_i$ for $|i-j|>1$

3. $E_i E_{i+1} E_i = E_i$



<u>Fact</u> This is semi-simple when q is not a root of 1.

$\exists$ inclusion $TL_n \hookrightarrow TL_{n+1}$ by adding one string on the right.

Describe structure of TL_n using a Bratteli diagram

$$B = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$$

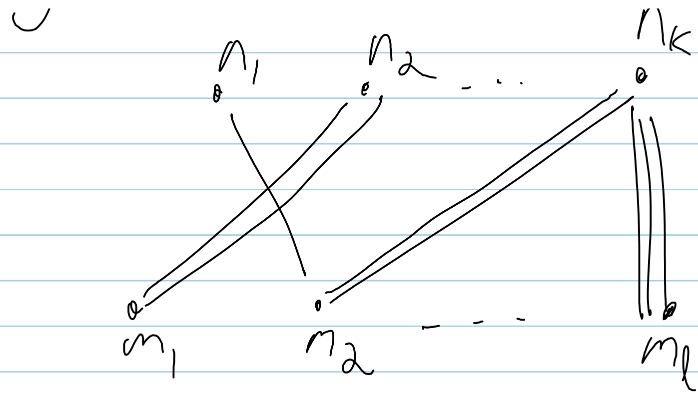
$$\cup$$

$$A = \bigoplus_{j=1}^{l} M_{m_j}(\mathbb{C})$$

each component of the map must be 0 or Id.

The diagram:

n n n_k



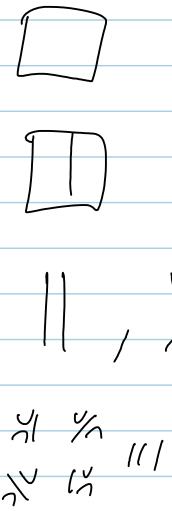
when q isn't $\sqrt{1}$, get

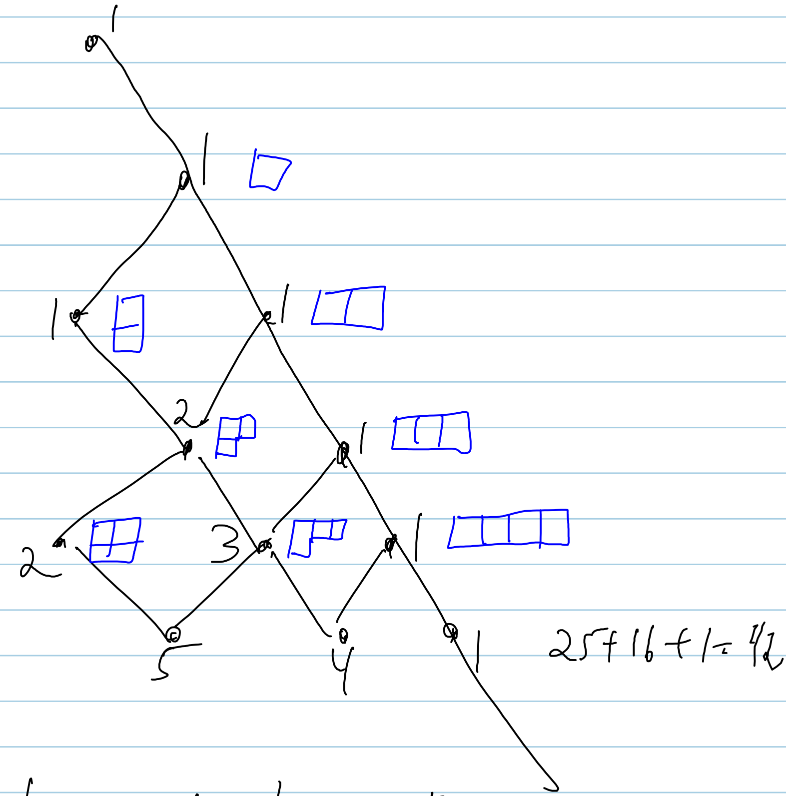
TL_0

TL_1

TL_2

TL_3





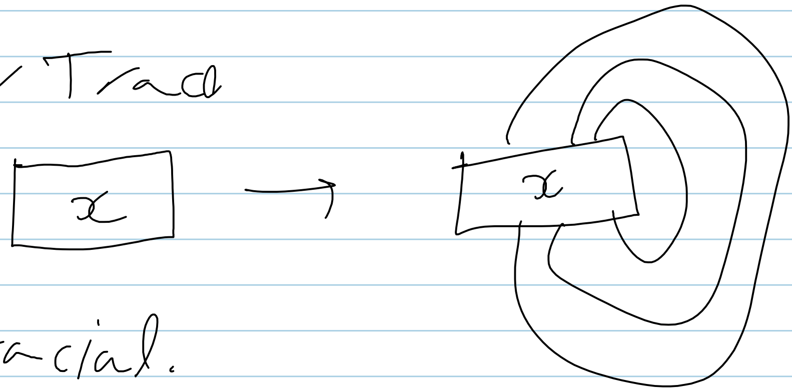
Note to self: Time to understand the

Artin–Wedderburn theorem

Pasted from <http://en.wikipedia.org/wiki/Artin%E2%80%93Wedderburn_theorem>

Positivity: TL has an involution
— a period 2 contravariant functor,
given by vertical reflection, and
conjugation of the coefficients.

Markov Trace



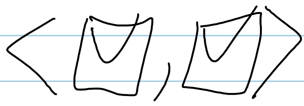
it is tracial.

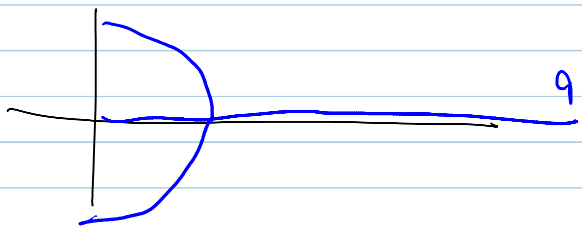
$\Rightarrow$ get a sesquilinear form on $TL(m,n)$

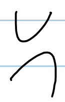
$$\langle x, y \rangle \longmapsto tr(y^* x)$$

Thm (Jones): $\langle\, ;\, \rangle$ is positive semi-def for all m,n iff $[2]_q \in \{2\cos^2\frac{\pi}{n} \mid n \geq 3\} \cup [2,\infty)$

So $q = e^{2\pi i/2n}$ for $n \geq 3$ or $q \geq 1$.

Example  $= q + q^{-1} \geq 0$

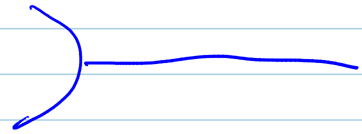
$\Rightarrow$ $q \in$ 

$f^{(2)} = || - \frac{1}{[2]}$ 

$$0 \leq \langle f^{(2)}, f^{(2)} \rangle = [3] = q^2 + 1 + q^{-2}$$

$$= (q + q^{-1})^2 - 1 \ldots$$

So $q + q^2 \geq 1$ so

$q \in$ 

More at http://drorbn.net/AcademicPensieve/2012-08/nb/TL%20Eigenvalues.pdf