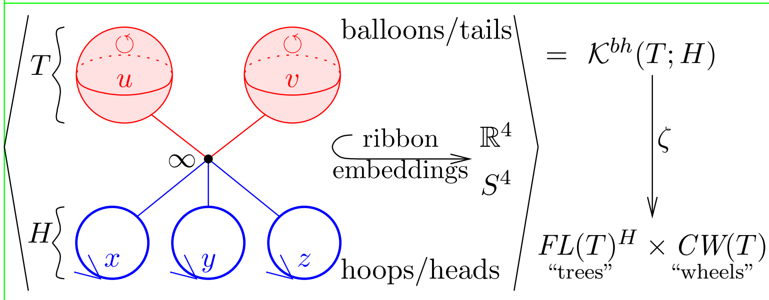
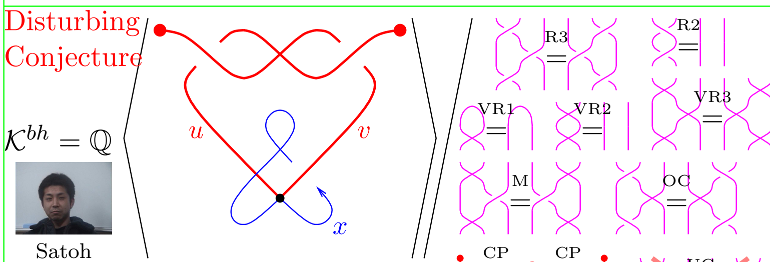
Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.

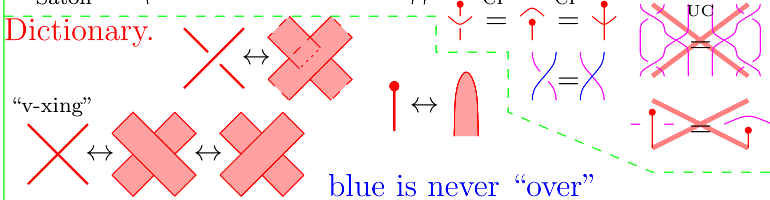


My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

Disturbing Conjecture



Dictionary.



Expansions the semi-virtual

Let $\mathcal{I}^n :=$ ⟨pictures with $\geq n$ semi-virts⟩ $\subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z\colon \mathcal{K}^{bh} \to \operatorname{gr}\mathcal{K}^{bh} = \widehat{\bigoplus}\mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

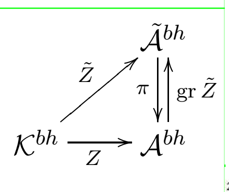
satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

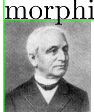
$$Z(\gamma) = (0,\ldots,0,\gamma/\mathcal{I}^{n+1},*,*,\ldots).$$

X.-S. Lin

Why? • Just because, and this is vastly more general.
• $\left(\mathcal{K}^{bh}/\mathcal{I}^{n+1}\right)^\star$ is "finite-type/polynomial invariants".
• The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K}\colon f(0)=0\}$. Then $\mathcal{I}^n = \{f\colon f$ vanishes like $|x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi\colon \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z}\colon \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi \,/\!\!/\, \operatorname{gr}\tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} \,/\!\!/\, \pi$ is an expansion.

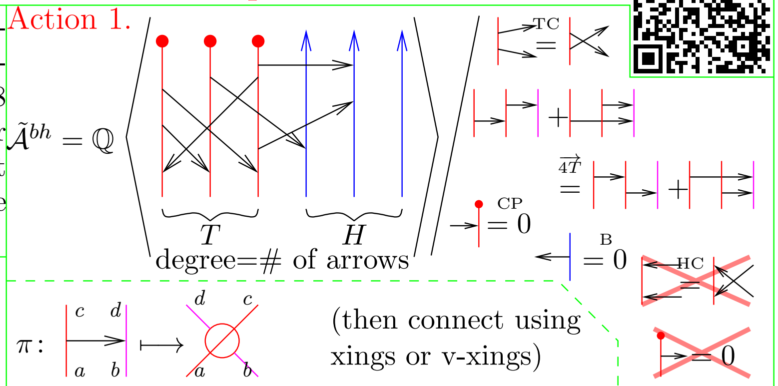




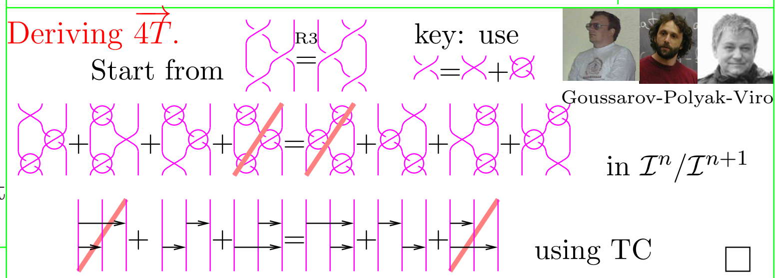
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

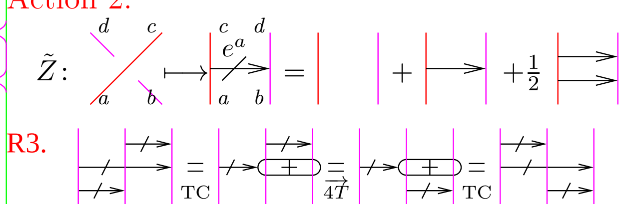
Action 1.



Deriving $\overrightarrow{4T}$. Start from

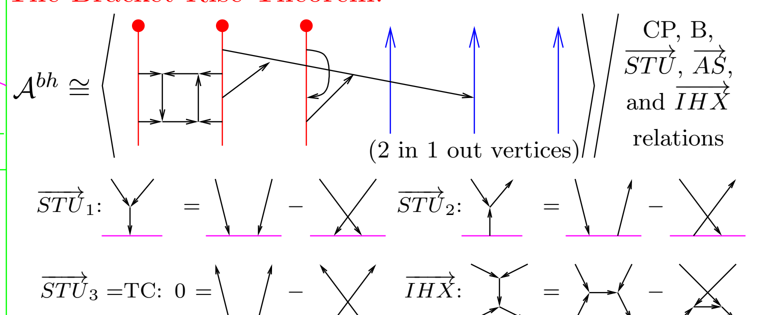


Action 2.

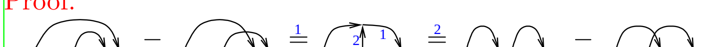


Exercise. Prove property U.

The Bracket-Rise Theorem.



Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5: