Meta-Groups, Meta-Bicrossed Products and the Alexander Polynomial

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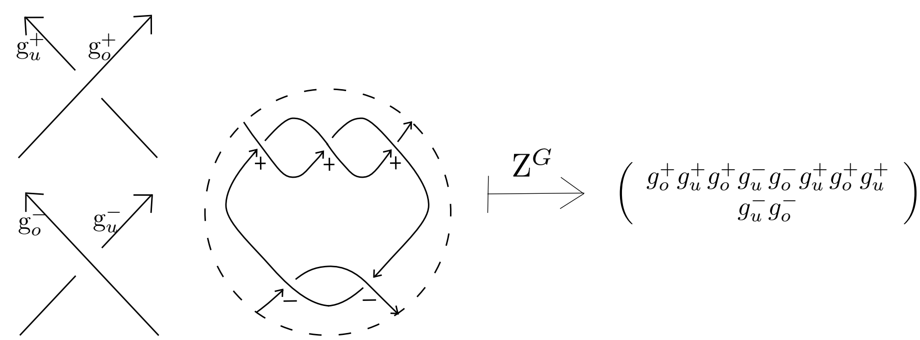
Abstract

Later.

Dror story here. B:

1 Warm-up: the baby invariant, Z^G

Let T be an oriented tangle diagram. Let G be a group, and suppose we are given two pairs $R^{\pm} = (g_o^{\pm}, g_u^{\pm})$ of elements of G. At each positive (resp. negative)[1] crossing of T, assign g_o^+ (resp. g_o^-) to the upper strand and g_u^+ (resp. g_u^-) to the lower strand, as in Figure 1. Then, for every strand, multiply all elements assigned to it in the order that they appear and store the end result. If T has n strands, we get a collection of n elements of G. Call this collection $Z^G(T)$.



(a) assigning values to crossings

(b) collecting along strands

Figure 1: Computing Z^G of a tangle } change font to Dror's convention.

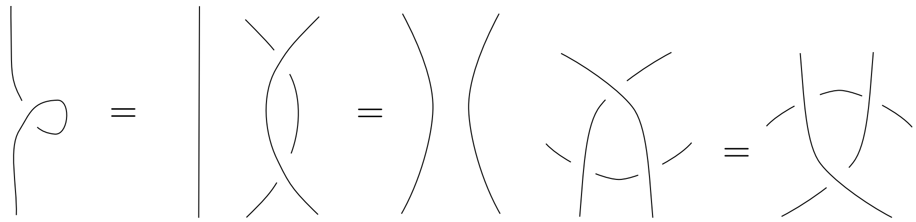
Unfortunately, the gods are not so kind and Z^G is A not worth much more than the effort that went in it. Indeed, invariance under the Reidemeister *II* move demands $g_o^- = (g_o^+)^{-1}$ and $g_u^- = (g_u^+)^{-1}$, while Reidemeister *III* adds that g_o^+ and g_u^+, as well as g_o^- and g_u^-, commute. As a result, every component of $Z^G(T)$ collapses to the form $g_o^a g_u^b$ for some integers a and b, so all the information to bring home is the sign number of times a given strand crosses over or under any other strand. It will turn out, nevertheless, that a similar procedure yields an amply non-trivial invariant with novel properties.

[1] Signs are determined by the "right-hand rule": If the right-hand thumb points along the direction of the upper strand of a positive crossing, then the fingers curl in the direction of the lower strand.

A: see Figure 2.

B: these are lecture notes for talks given by the first author, written and completed by the second. The talks, with handouts and videos, are available online at [GWU URL] and [Regina URL]. See also further

Comments at [Caen VRL, June 8 clips].



(a) Reidemeister I (b) Reidemeister II (c) Reidemeister III

Figure 2: The three Reidemeister moves

2 A better invariant: Z^β

The invariant that we wish to introduce can be thought of as taking values in a meta-group. This is a generalization of what we call a "group computer":

2.1 Preliminary: A Group Computer

If X is a finite set and G is a group we let G^X denote the set of all possible assignments of elements of G to the set X.; these are "G-valued datasets, with registers labelled by the elements of X". A "group computer" can manipulate such datasets in some prescribed ways:

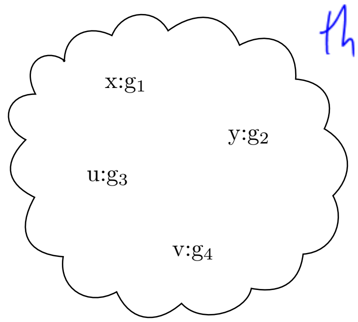


Figure 3: A typical element of $G^{\{x,y,u,v\}}$

~~We use the operations of G to implement operations on the computer G^X in the obvious way.~~ For example, define[2] $m^z_{xy} : G^{\{x,y\}} \to G^{\{z\}}$ using the group multiplication, $\{x : g_1, y : g_2\} \mapsto \{z : g_1 g_2\}$. There are obvious unary operations for inversion, doubling, renaming, deletion and unit insertion, respectively denoted S^x, Δ^x_{yz}, ρ^y_x, d_x and e^y, and respectively implemented on $G^{\{x\}}$ as $\{x : g\}$ mapping to $\{x : g^{-1}\}$, $\{y : g, z : g\}$, $\{y : g\}$, $\{\}$ and $\{x : g, y : e\}$. In addition there is a binary operation for merging data sets, $\bigcup : G^X \times G^Y \to G^{X \cup Y}$, which takes two data sets P and Q and makes their disjoint union $P \cup Q$.

cloning an element renaming a register

2.2 Meta-Groups

The operations on a group computer obey a certain set of basic set-theoretic axioms as well as axioms inherited from the group G. A meta-group is an abstract computer that satisfies some but not all of those axioms. We postpone the precise definition to Section 3; it is best to begin with a prototypical example, as follows. Let $G_X := M_{X \times X}(\mathbf{Z})$ denote (not in reference to any group G) the set of $|X| \times |X|$ matrices of integers with rows and columns labelled by X. The operation of "multiplication", on say, 3×3 matrices,

[2]To avoid cluttering notation, we write $\{x, y\}$ for a set containing x and y, and possibly more elements. We also assume common sense with respect to naming: for instance, if an operation creates the register x, we assume it did not exist before.

$m_z^{xy} : G_{\{x,y,w\}} \to G_{\{z,w\}}$, is defined by simultaneously adding rows and columns labelled by x and y:

$$\begin{array}{c} \\ x \\ y \\ w \end{array} \begin{array}{c} \begin{array}{ccc} x & y & w \end{array} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \end{array} \mapsto \begin{array}{c} \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cc} z & w \end{array} \\ \begin{pmatrix} a+b+d+e & c+f \\ g+h & i \end{pmatrix} \end{array}$$

also state that this example is still "associtive"?

One critical way in which this example differs from a group computer is the failure of the axiom

$$\cancel{d_y G_{\{x,y\}} \cup d_x G_{\{x,y\}} = \mathrm{id}(G_{\{x,y\}})}$$

$d_y P \cup d_x P = P$, for $P \in G_{\{x,y\}}$

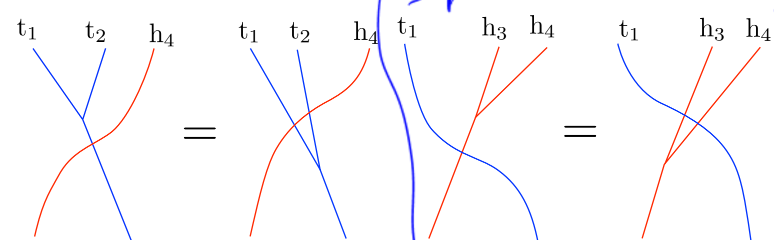
Indeed, if $P \in G_{\{x,y\}}$ is the matrix $\begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$, then

$$d_y P \cup d_x P = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \end{array} \neq P$$

2.3 Meta-Bicrossed Products

Suppose a group G is given as the product $G = TH$ of two of its subgroups, where $T \cap H = \{e\}$. Then also $G = HT$[2] and every element of G has unique[3] representations of the form th and $h't'$ where $h, h' \in H$ and $t, t' \in T$. Accordingly there is a "swap" map $sw : T \times H \to H \times T$, $(t,h) \mapsto (h',t')$ such that if $g = th$ then $g = h't'$ also. The swap map satisfies some relations; in group-computer language, the important ones read

separate



font & numbering

what's // ?

(a) $tm_1^{12} /\!\!/ sw_{14} = sw_{24} /\!\!/ sw_{14} /\!\!/ tm_1^{12}$ (b) $hm_3^{34} /\!\!/ sw_{13} = sw_{13} /\!\!/ sw_{14} /\!\!/ hm_3^{34}$

where tm and hm stand for multiplication in T and H respectively. Conversely the data (H, T, sw) determines a group G, called the bicrossed product of H and T, provided that sw satisfies relations (a) and (b). ← + a few minor relations.

The corresponding notion of a meta-bicrossed product is a collection of sets $\beta(H,T)$ indexed by all *pairs* of finite sets H and T, and equipped with multiplication maps tm_z^{xy}, hm_z^{xy} and a swap map sw_{xy}^{th} satisfying (a) and (b). A meta-bicrossed product defines a meta-group with $\Gamma_X = \beta(X,X)$. A prototypical example is again given by (now rectangular) matrices, $\mu(H,T) := M_{T\times H}(\mathbf{Z})$, with tm_z^{xy} and hm_z^{xy} corresponding to adding two rows and adding two columns, and swap being the trivial operation:

you have to be more careful here about which index lies in which set.

How?

[3]Separation of variables: suppose $g = h_1t_1 = h_2t_2$. Then we have $h_2^{-1}h_1 = t_2t_1^{-1}$, which implies that $h_1 = h_2$ and $t_1 = t_2$ since $h_2^{-1}h_1 \in H$, $t_2t_1^{-1} \in T$, and $H \cap T = \{e\}$

Foot 2: Indeed, if $g^{-1} = th$ then $g = h^{-1}t^{-1}$, so $g^{-1} \in TH$ implies $g \in HT$, and as $TH = G$, also $HT = G$.

$$\begin{array}{cc} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array} \xmapsto{hm_1^{1,2}} \begin{array}{cc} & h_1 \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} a+b \\ c+d \\ e+f \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array} \xmapsto{tm_1^{1,2}} \begin{array}{cc} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} t_1 \\ t_3 \end{array} & \begin{pmatrix} a+c & b+d \\ e & f \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array} \xmapsto{sw_{1,2}^{th}} \begin{array}{cc} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array}$$

needs better alignment!

2.4 β Calculus

The β calculus has an arcane origin [VIDEO] which is not appropriate to mention here. We expect that it can be presented in a much simpler and fitting context than that in which it was discovered. Accordingly we will simply pull it out of a hat. Let $\beta(H,T)$ be (again, in reference to *sets* H and T) the collection of arrays with rows labeled by $t_i \in T$ and columns labeled by $h_j \in H$, along with a distinguished element ω. Such arrays are conveniently presented in the following format

$$\begin{array}{c|ccc} \omega & h_1 & h_2 & \dots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array}$$

The α_{ij} and ω are rational functions of variables T_i, which are in bijection with the row labels t_i.

$\beta(H,T)$ is equipped with a peculiar set of operations. Despite being repulsive at sight, they are completely elementary. They are defined as follows

$$tm_z^{xy} : \begin{array}{c|c} \omega & \dots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \dots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}$$

(here α & β are rows & γ is a matrix)

In tm there is also a change of variables, $T_x, T_y \to T_z$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & h_x & h_y & \dots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \dots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array}$$

(here . . .)

$$sw_{xy}^{th} : \begin{array}{c|cc} \omega & h_y & \dots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \dots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array}$$

(here . . .)

Here $\epsilon = 1+\alpha$, $\langle\alpha\rangle = \sum_i \alpha_i$, and $\langle\gamma\rangle = \sum_{i\neq x}\gamma_i$. Note also that in sw_{xy}^{th}, the term $\gamma\beta$ is the matrix product of the column γ with the row β and hence has the same dimensions as the matrix δ. We also need the disjoint union, defined by

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array}$$

Finally there are two elements which will serve as a pair of "R-matrices", analogous to the pairs $(g_o^\pm, g_u^\pm)$ of Z^G:

$$R^+_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x - 1 \\ t_y & 0 & 0 \end{array} \qquad R^-_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x^{-1} - 1 \\ t_y & 0 & 0 \end{array}$$

2.5 Z^β

Let T be again an oriented tangle diagram. At each crossing, assign a number to the upper strand and to the lower strand. Form the disjoint union $\bigcup_{\{i,j\}} R^\pm_{ij}$ where $\{i,j\}$ runs over all pairs assigned to crossings, with i labelling the upper strand and j labelling the lower strand, and where $\pm$ is determined by the sign of the given crossing. Now for each strand multiply all the labels in the order in which they appear. That is, if the first label on the strand is k, apply $gm_k^{kl} = sw_{kl} \,/\!/\, tm_k^{kl} \,/\!/\, hm_k^{kl}$ where l runs over all labels subsequently encountered on the strand (in order). If T has n strands, the result is an $n \times n$ array with corner element. Call this array $Z^\beta(T)$.

a

As an example, for the knot 8^{17} [ROLFSEN] illustrated in Figure 4, make the disjoint union[4] $R^-_{12,1}R^-_{2,7}R^-_{8,3}R^-_{4,11}R^+_{16,5}R^+_{6,13}R$ which is given by the following array:

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overflow

1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$T_2^{-1}-1$	0	0	0	0
t_4	0	0	0	0	0	$T_4^{-1}-1$	0	0
t_6	0	0	0	0	0	0	T_6-1	0
t_8	0	$T_8^{-1}-1$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$T_{10}-1$
t_{12}	$T_{12}^{-1}-1$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$T_{14}-1$	0	0	0
t_{16}	0	0	$T_{16}-1$	0	0	0	0	0

empty rows & columns are omitted

Then apply the multiplications $gm^1_{1,k}$, with k running from 1 to 16, to get the following 1×1 array with corner element:

$$\begin{array}{c|c} -T_1^{-3} + 4T_1^{-2} - 8T_1^{-1} + 11 - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ \hline t_1 & 0 \end{array}$$

Theorem 1. *Z^β is an invariant of oriented tangle diagrams. Restricted to knots, the corner element is the Alexander polynomial. Restricted to braids, it is equivalent to the Burau representation.*

Proof. Later. □

One philosophically appealing major property of Z^β is that the operations used to compute it have a literal interpretation of gluing crossings together. In particular, at every stage of the computation we get an invariant of the tangle[5] made of all the crossings but only those for which the corresponding gm was carried out have been glued.

space

[4] From now on we omit the $\cup$ in disjoint unions: $\beta_1\beta_2 := \beta_1 \cup \beta_2$. We also suppress rows/columns of zeros.

[5] the careful reader may wish to peek ahead at Section 3.1 for a better grasp of this statement

side by side to save space?

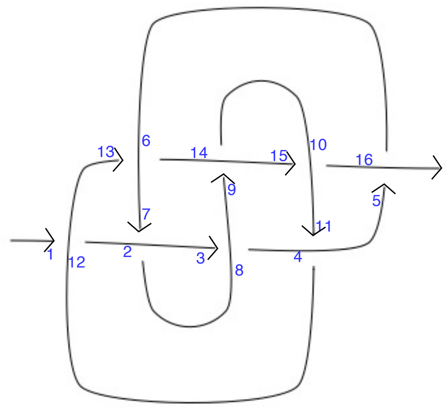

Figure 4: The knot 8^{17} with crossings labelled

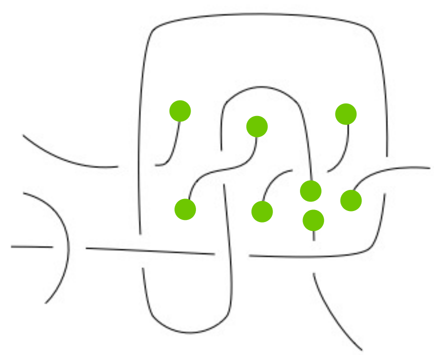

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Figure 5: 8^{17} after attaching crossings 1 through 10. The arcs with green dots can not make it out to the boundary disk.

3 More on meta-groups

3.1 The meta-group of coloured v-tangles

When one tries to follow the interpretation of the computation of Z^β as progressively attaching crossings together to form a tangle, one will in general encounter a step where the tangle will become non-planar (a strand will have to go through another in an "artificial" crossing to reach the boundary disk). See Figure 5. Such tangles are called virtual or v-tangles and constitute a rich subject of study on their own. For us it will suffice to give them a name. A

The operation

Armed with this new word in our vocabulary we can now define what seems to be the "most natural" meta-group: the meta-group of oriented coloured v-tangles. Let Γ_X be the set of v-tangles with strands labelled by X. There is a natural definition for all the meta-group operations. m_z^{xy} [6] concatenates strand x with strand y and labels the resulting strand z (note that we *need* virtual tangles for this to be well-defined), S^x reverses the orientation of strand x, e^x creates an isolated strand with label x, d^x deletes strand x, and Δ^x_{yz} is the cabling operation with input strand x and output strands y and z. [EXPAND]

[6]Remark: this is *not* a meta-generalization of the group structure on braids.

A: philosophy must be followed by a formal definition, or at least a link to a formal definition.

3.2 Some precise definitions

Axioms lists and checks.

3.3 More examples

Examples 1,2,3,4,5,6,7, relationship to example ∞.

4 Towards Categorification

4.1 Difficulties with classical Alexander

4.2 Removing denominators