

Meta-Groups, Meta-Bicrossed-Products, and the Alexander Polynomial, 1
 Dror Bar-Natan in Regina, June 2012
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Abstract. The a priori expectation of first year elementary school students who were just introduced to the natural numbers, if they would be ready to verbalize it, must be that soon, perhaps by second grade, they'd master the theory and know all there is to know about those numbers. But they would be wrong, for number theory remains a thriving subject, well-connected to practically anything there is out there in mathematics.

I was a bit more sophisticated when I first heard of knot theory. My first thought was that it was either trivial or intractable, and most definitely, I wasn't going to learn it is interesting. But it is, and I was wrong, for the reader of knot theory is often lead to the most interesting and beautiful structures in topology, geometry, quantum field theory, and algebra.

Today I will talk about just one minor example, mostly having to do with the link to algebra: A straightforward proposal for a group-theoretic invariant of knots fails if one really means groups, but works once generalized to meta-groups (to be defined). We will construct one complicated but elementary meta-group as a meta-bicrossed-product (to be defined), and explain how the resulting invariant is a not-yet-understood yet potentially significant generalization of the Alexander polynomial, while at the same time being a specialization of a somewhat-understood "universal finite type invariant of w-knots" and of an elusive "universal finite type invariant of v-knots".

Alexander Issues.

- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost. (e.g. Archibald's thesis)
- Hard to categorify.

Jana Archibald.

Idea. Given a group G and two "YB" pairs $R^\pm = (g_a^\pm, g_b^\pm) \in G^2$, map them to crossings and "multiply along", so that

This Fails! R2 implies that $g_a^\pm g_b^\pm = c = g_b^\pm g_a^\pm$ and then R3 implies that g_a^\pm and g_b^\pm commute, so the result is a simple counting invariant.

A Group Computer. Given G , can store group elements and perform operations on them:

Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ_{xy}^\pm for element cloning, ρ_y^\pm for remainings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those.

A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ , and a collection of operations $m_x^{xy}, S_x, e_x, d_x, \Delta_{xy}^\pm$ (sometimes), ρ_y^\pm , and \cup , satisfying the exact same linear properties.

Example 1. The non-meta example, $G_\gamma := G^\gamma$.

Example 2. $G_\gamma := M_{\gamma \times \gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H, T , and the "swap" map $sw^{th} : (t, h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw : T \times H \rightarrow H \times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the "bicrossed product".

A Standard Alexander Formula. Label the arcs 1 through $(n+1) = 1$, make an $n \times n$ matrix as below, delete one row and one column, and compute the determinant:

$$\begin{pmatrix} a & b & c \\ c & -1 & -X & X \\ a & b & c \\ c & -X & X-1 & 1 \end{pmatrix}$$

[[1 :: 7, 1 :: 7]] // Det

$$\begin{pmatrix} -1 & X & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ 0 & -1 & X & 0 & 1 & -X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X & 0 \\ 0 & 0 & 1 & -X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & 0 & -X & 1 \end{pmatrix}$$

$-1 + 4X - 8X^2 + 11X^3 - 8X^4 + 4X^5 - X^6$

(e.g. Archibald's thesis)

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Note:
 $P \neq d_y(P) \cup d_x(T)$

Note: A meta-group plus a YB pair give a knot/tangle invariant.

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Where does it come from? The accidental¹ answer is that it is a symbolic calculus for a natural reduction⁴ of the unique homomorphic expansion² of w-tangles³.

1. "Accidental" for it's only how I came about it. There ought to be a better answer.
2. A "homomorphic expansion", aka as a homomorphic universal finite type invariant, is a completely canonical construct whose presence implies that the objects in questions are susceptible to study using graded algebra.
3. "v-Tangles" are the meta-group generated by crossings modulo Reidemeister moves. "w-Tangles" are a natural quotient of v-tangles. They are at least related and perhaps identical to a certain class of 1D/2D knots in 4D.
4. To "only what is visible by the 2D Lie algebra".

A certain generalization will arise by not reducing as in 4. A vast generalization may arise when homomorphic expansions for v-tangles are understood, a task likely equivalent to the Etingof-Kazhdan quantization of Lie bialgebras.

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<< KnotTheory
Alexander[Knot[8, 17]][X] // Factor
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Read more at http://katlas.org/wiki/KnotTheory.
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$$\frac{1-4x-8x^2-11x^3+8x^4-4x^5-x^6}{x^3}$$

The key trick: $\frac{\omega}{t_i} \Big| \frac{h_j}{\alpha_{ij}} \longleftrightarrow B(\omega, \Lambda = \sum_{i,j} \alpha_{ij} t_i h_j).$