Lyndon Words

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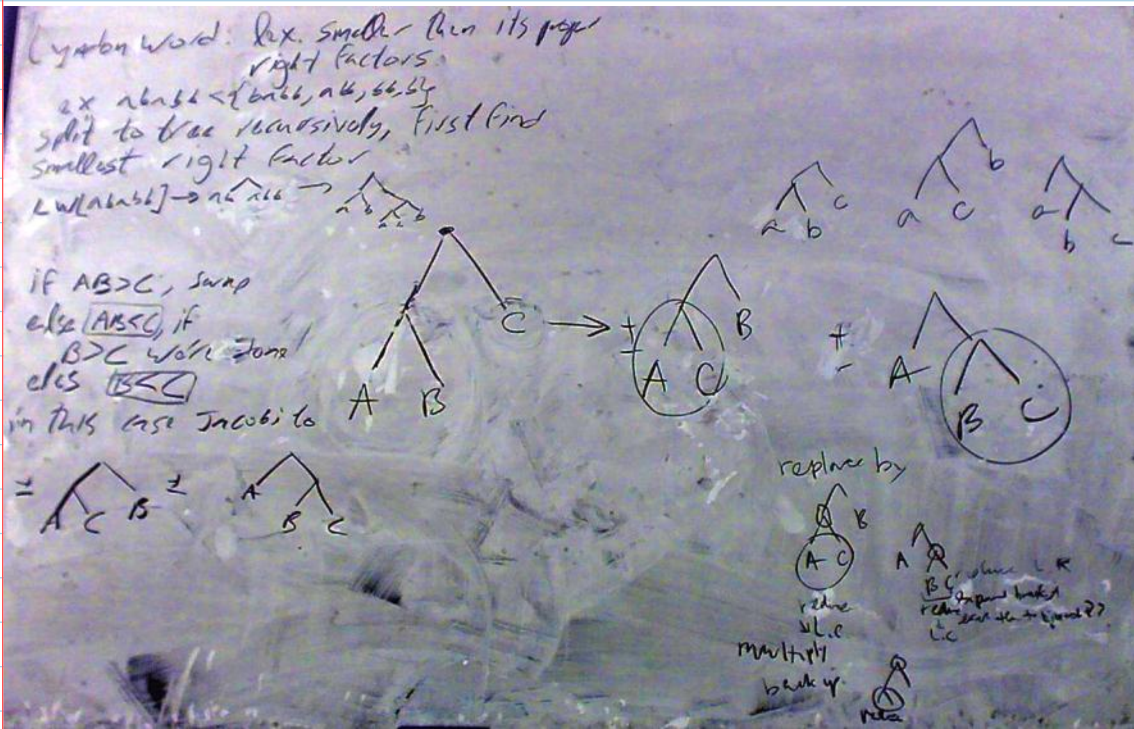
http://katlas.math.toronto.edu/drorbn/bbs/show?shot=Chu-071214-182203.jpg:



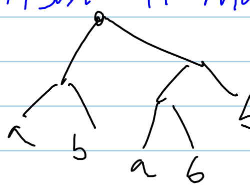
Lyndon word: Lexicographically smaller than its proper right subwords.

Example: ababb < {babb, abb, bb, b}

Can split to a tree recursively - First find smallest right factor: [in itself it must be Lyndon]

$l(ababb) \longrightarrow ab\widehat{\ }abb \longrightarrow$

Lyndon word



Lyndon element.

Every element in FreeLie is a combination of Lyndon elements:

Take [[A,B],C] where AB & C are Lyndon.

$$\underbrace{\qquad\qquad\qquad}_{A}\ \underbrace{\qquad\qquad}_{B}\ \underbrace{\qquad\qquad}_{C}$$

If $C < AB$, swap
else $AB < C$ then ABC is Lyndon
if $B \geq C$ then $\tau(ABC) = [\tau(AB), \tau(C)]$
otherwise $A < B < C$; then ACB is also Lyndon
use Jacobi to write
$[[A, B], C] = [[A, C], B] + [A, [B, C]]$

Claim. The bracket of two Lyndon elements is a combination of Lyndon elements greater or equal to the smaller of their two concatanations.

Proof. WLOG, we are looking at $[\tau(L), \tau(R)]$ where $L < R$, otherwise use AS. Clearly, LR is Lyndon.

We then argue by induction on the total degree, and then on the lexicographical ordering on LR, starting from the end and going down.
If L is atomic then R is the minimal right factor of LR so $\tau(LR) = [\tau(L), \tau(R)]$ and all is well.
Therwise $L = AB$, $R = C$, with $AB < C$, and with B the minimal proper right factor of AB.
If $B \geq C$, then C is the smallest right

factor of ABC so

$$\eta(ABC) = [\eta(AB), \eta(C)] = [\eta(L), \eta(R)]$$

and all is well.

Otherwise $A < B < C$ and we have

$$[\eta(L), \eta(R)] = [[\eta A, \eta B], \eta C] = \quad \text{using Jacobi}$$

$$= [\eta A, \underset{1}{[\eta B}, \eta C]] + [[\eta A, \underset{2}{\eta C}], \eta B]$$

$= \eta(ABC)$ + higher terms (from 1)

+ further higher terms (from 2).