Classification of Flat Virtual Pure Tangles and Basis for its Associated Graded Algebra

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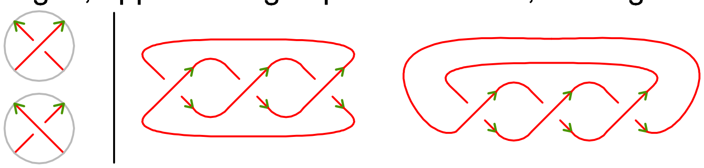
Why do we care about virtual knots?

- Virtual Knots generalize classical knots; in fact, classical knots inject into virtual knots.
- Any quantum invariant for classical knots extend to virtual knots (a variant), so virtual knots may be a more natural domain for quantum invariants.

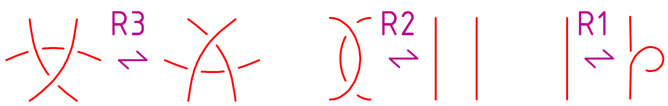
Definition - Virtual Knots

Usual Knots diagrams: *Planar* directed graphs with only *crossings.*

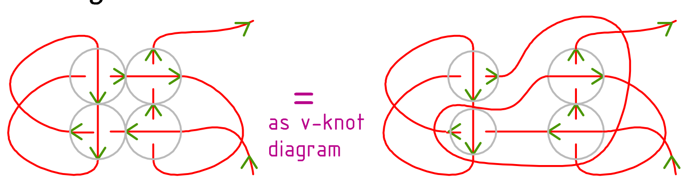
Crossings: tetravalent vertex with cyclically ordered edges, opposite edges paired ordered, and signed



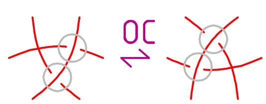
Relations:



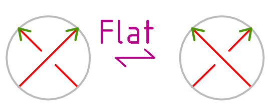
Virtual Knot Diagrams: **Directed** graphs with only *crossings* as vertices.



w:



Flat:

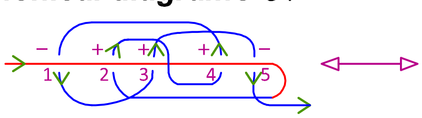


Our Subject: Flat Virtual *long knot*, i.e. the *skeleton* is a line.

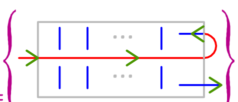
Theorem 1 [C./conjectured by Bar-Natan]

(Classification of Flat Virtual Long Knots, $f\mathcal{K}$)

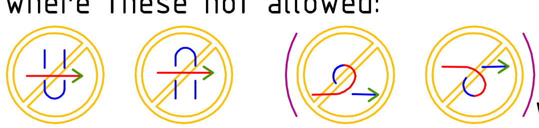
Flat virtual long knots are in bijection with the canonical diagrams $\mathcal{C}$.



σ: $\sigma(1) = (3, -)$, $\sigma(2) = (1, +)$, $\sigma(3) = (4, +)$, $\sigma(4) = (2, +)$, $\sigma(5) = (5, +)$

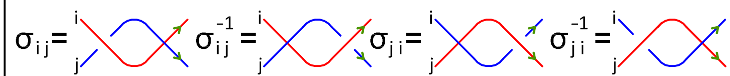
$f\mathcal{K} \overset{1-1}{\leftrightarrow} \mathcal{C} := \{$  $\} \overset{1-1}{\leftrightarrow} \{\sigma : \{1..n\} \to \{1..n\} \times \{+,-\}$ s.t. $\pi_1 \circ \sigma \in S_n$, $(\sigma(i), \sigma(i+1)) \neq ((j,\pm), (j+1,\mp))$ or $((j+1,+), (j,\mp))\}$

where these not allowed:



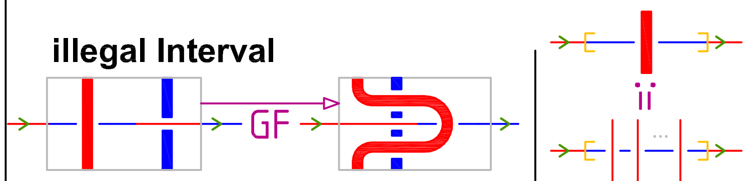
Remarks:

- generalizes easily to multiple (labelled) strands (pure tangles).
- Invariants for virtual long knots/pure tangles.
- Topological: immersed curves on surfaces with a circle boundary where curves end on the boundary modulo adding and removing empty handles/
- Invariant for the flat virtual braid group $\langle \sigma_{ij} \mid \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij}, \sigma_{ij} \leftrightarrow \sigma_{kl}, \sigma_{ij} = \sigma_{ji}^{-1} \rangle$
  Conjecture: flat braids inject into flat tangles.



Sketch of Proof Thm 1: 2-dim Induction.

- There is a 2-d parameter space (# illegal intervals, # crossings) such that sorting moves would move in a "lowering direction."



- the non-trivial part: show that different sequences of sorting moves lead to the same result by using relations between relations (see big diagram next page) in the diamond lemma.
- easy: well-defined under all R-moves.

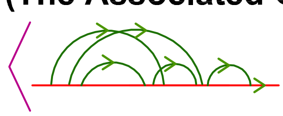
Which "Associated Graded"?

The target space of a *universal finite-type invariant,* the one associated to the filtration by (generalized) *powers* of the (generalized) *augmentation ideal*.

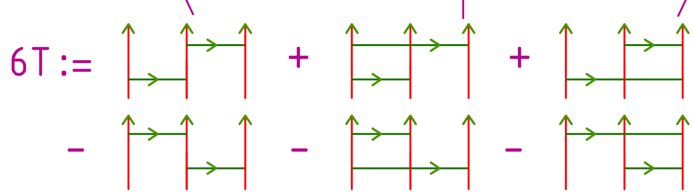
Why do we care?

For knots, v-knots, these are related to Lie-algebras: given any finite-dim Lie (bi-) algebra, there is a map from these into tensors of the enveloping algebras.

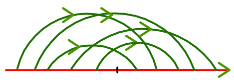
Definition (The Associated Graded Space)

$\mathcal{A}^{fb} := \langle$  $\mid$ 6T (from R3) $\rangle$

6T :=



Main Theorem 2 [C., conj. by Bar-Natan]
Basis for the associated graded spaces

Basis $\mathcal{A}^{fb}$ = {  } $\overset{1-1}{\longleftrightarrow}$ { (1,2,3,4,5) $\mapsto$ (2,4,1,5,3) }

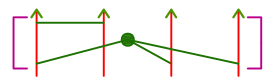
Remarks

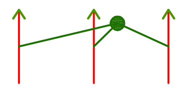
- dim $\mathcal{A}^{fb}$=n!.
- generalizes to n-strand skeleton.
- Variants of flat virtual knots: {All R23} / {braid-like R23}, R1/R1 and we obtain basis for their associated graded spaces
- $\mathcal{A}^{fb}$, and the other variants, for fixed *n* are in particular associative algebras; and for all n, there isa richer gluing structure.
- A generalized Grobner basis for the horizontal algebra, which is the enveloping algebra of some Lie algebra, while a usual Grobner basis could not be found.

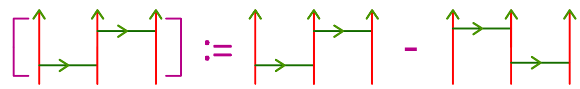
Idea of Proof for Main Theorem 2:
Gröbner basis for Chord Diagram Algebra

Generalize Grobner basis for associative algebras to chord diagrams algebras with the gluing structure. The main idea is to define a partial ordering on diagrams and require that it to be respected by the gluing structure, and have unique leading terms in relations and syzygies.

A Syzygy (relation of relation):

6T-6T : $\sum$ [] = 0

where 6T : 



and the sum is over 4 ways of placing 6T and for each way 3 ways of placing one end of the arrow on the strands touched by the 6T.

Relation between the Group-Finger relations: