Khovanskii: How to Solve a Cubic Equation?

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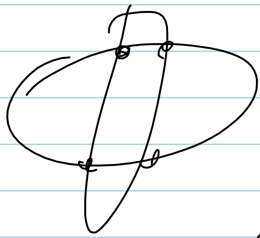
1. Given $a = \cos x$, solve for $\frac{x}{n}$

... at $n=3$ this turns out to solve the general cubic.

2. Given two quadrics, find their point of intersections



... This turns out to solve the general quartic.

$\cos nx = P_n(\cos x)$ the Chebychev poly.

now solve $P_n(x) = a$?

$z = \cos x + i \sin x \qquad z^n = \cos nx + i \sin nx$

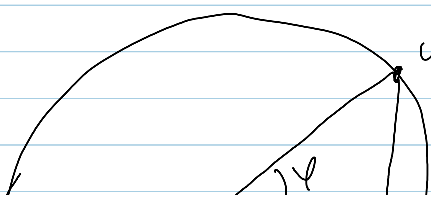
So $\cos nx = \mathrm{Re}(\cos x + i \sin x)^n$

$= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \ldots$

$=$

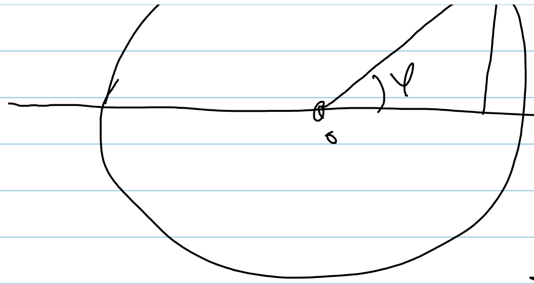
So $P_n(y) = y^n - \binom{n}{2} y^{n-2}(1-y^2) + \binom{n}{4} y^{n-4}(1-y^2)^2 - \ldots$

So $P_3(y) = y^3 - 3y(1-y^2) = 4y^2 - 3y$



$u = x + iy, \ \|u\| = 1$

$x = \frac{u + u^{-1}}{2}$



$x = \frac{u+u^{-1}}{2}$

$a = \cos n\psi = \frac{u^n + u^{-n}}{2}$

$\longrightarrow$ this is a quadratic eq'n for u^n

$$\Rightarrow (u^n)^2 - 2a u^n + 1 = 0$$

--- find u_1, u_2 by taking nth roots,

--- find $x = \frac{u+u^{-1}}{2}$

I.e. if $y = \frac{u+u^{-1}}{2}$, $P_3(y) = \frac{u^3+u^{-3}}{2} = a$

is easily soluble.

If we want to solve

$$x^n + a_{n-1}x^{n-1} + \dots a_0 = 0$$

by a translation, can kill a_{n-1}: take

$x = y - \frac{a_{n-1}}{n}$, get

$$y^n + b_{n-2}y^{n-2} + \dots = 0$$

In the cubic case we get

$$x^3 + px + q = 0$$

now rescale: $x = \lambda y$:

$$c\lambda^3 y^3 + pc\lambda x + qc = 0$$

Solve $c\lambda^2=4$, $c\lambda=3$ & get to the "Chebychev eq'n".

Over a general field, w/ char $\neq 2$, P_n is defined by

$$P_n\left(\frac{u+u^{-1}}{2}\right)=\frac{u^n+u^{-n}}{2}$$

Let $x(u)=\frac{u+u^{-1}}{2}$ $\quad y(u)=\frac{u-u^{-1}}{2i}$

Then $u=x(u)+iy(u)$; $x^2(u)+y^2(u)=1$

So this is a general analogue of cos, sin.

$P_n:\mathbb{C}\longrightarrow\mathbb{C}$ satisfies $P_k\circ P_n=P_{nk}$

The dynamics of iterating P_n is funny;

For $x\notin[-1,1]$, always go to ∞.

For $x\in[-1,1]$, iterations stay in $[-1,1]$ but are chaotic there.

Now to the quartic:

$$P=a_1x^2+b_1xy+c_1y^2+d_1xz+e_1yz+f_1z^2=0$$
$$Q=a_2x^2+\ \_\ \ \_\ \ \_\ \ \_\ +f_2z^2=0$$

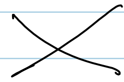
alternatively, take symmetric matrices $\tilde{P},\tilde{Q}$ and solve

$$(x\,y\,z)\tilde{P}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\ \ldots$$

[a cubic]

Find λ s.t. $\det(\tilde{P} + \lambda \tilde{Q}) = 0$ [a cubic eq'n]

now solve $P = 0$ 

$P + \lambda Q = 0$  degenerate.

So this we can solve!

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but now $ax^4 + bx^3 + cx^2 + dx + c = 0$

is equiv. to

$y = x^2$ & $ay^2 + bxy + cy + dx + e = 0$

which we already solved....

Topological Galois Theory

Polynomials.

$$y^n + a_n(x) y^{n-1} + \dots + a_n(x) = 0$$

Can the solution be written in terms of elementary functions (log, exp etc. included)

Theorem. Insoluble unless already algebraically soluble.