

Moral: Study the
"Bethe Ansatz"

Heisenberg spin chain - XXX model

(ref:
cond-mat/
9809162)

$$\hat{H} = 4 \sum_{l=1}^L \left(\frac{1}{4} - \vec{S}_l \cdot \vec{S}_{l+1} \right) \quad \text{w/ periodic boundary}$$

$$\hat{H} |\Psi\rangle = E |\Psi\rangle \quad |\Psi\rangle \in (\mathbb{C}^2)^{\otimes L}$$

Q: How important is it that the scalar part of H would be just what it is?

Rewrite: $\hat{H} = \sum_{l=1}^L \left(1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1} \right) =$

$$= \sum_{l=1}^L \left(1 - \sigma_l^3 \sigma_{l+1}^3 - 2\sigma_l^+ \sigma_{l+1}^- - 2\sigma_l^- \sigma_{l+1}^+ \right)$$

$$= 2 \sum_{l=1}^L \left(1 - P_{l,l+1} \right) = \sum X_{l,l+1}$$

Permutation; i.e. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Due to conservation of total spin, the

Hamiltonian decomposes into $\binom{L}{M} \times \binom{L}{M}$ blocks,

$$M = 0, \dots, L$$

Shift operator: $\hat{U} = P_{1,2} P_{2,3} \dots P_{L-1,L}$

$$[\hat{U}, \hat{H}] = 0, \quad [\hat{U}, \vec{S}] = 0 \quad \hat{U}^L = I$$

$$\Rightarrow \text{the eigenvalues } U = e^{\frac{2\pi i}{n} k}$$

Algebraic Solution: Introduce $u_k \in \mathbb{C}$ for each
"n.n" \downarrow " " \downarrow spin

Solve
$$\left(\frac{U_k + i/2}{U_k - i/2} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{U_k - U_j + i}{U_k - U_j - i}$$

Then

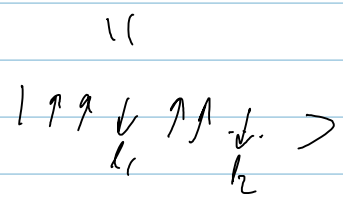
$$E = 2 \sum_{k=1}^M \frac{1}{u_k^2 + \frac{1}{4}} \quad U = \prod_{i=1}^M \frac{u_k + i/2}{u_k - i/2}$$

Proof 1 (coordinate Bethe ansatz, 1931)

write

$$|\Psi\rangle = \sum_{l_1 < \dots < l_M} \Psi(l_1, \dots, l_M) \underbrace{\bar{s}_{l_1} \bar{s}_{l_2} \dots \bar{s}_{l_M}}_{\downarrow} |0\rangle$$

w/ $|0\rangle = |\uparrow\uparrow\uparrow \dots \uparrow\rangle$



The Ansatz:

$$\Psi(l_1, \dots, l_M) = \sum_{\tau \in S_M} A(\tau) e^{i p_{\tau_1} l_1 + \dots + i p_{\tau_M} l_M}$$

Q: In the spirit of homology, can this image be written as a kernel?

So $|\downarrow\rangle$ is a "particle" called "magnon".

It works! w/

$$A(\tau) = \text{sign}(\tau) \prod_{j < k} (e^{i p_k + i p_j} - 2e^{i p_k} + 1)$$

$$E = \sum_{k=1}^M 8 \sin^2 \frac{p_k}{2} \quad U = \prod_{k=1}^M e^{i p_k}$$

On a finite lattice the p_k 's are quantized:

$$e^{i p_k L} = \prod_{j=1, j \neq k}^n S(p_k, p_j) \quad \leftarrow \text{The "S-"}$$

matrix

$$S(p_k, p_j) = - \frac{e^{i p_k + i p_j} - 2e^{i p_k} + 1}{e^{i p_k + i p_j} - 2e^{i p_j} + 1}$$

The S-matrix here is "factorized", only
2-particle interactions exist,

Nice change of variables: $e^{i p_k} = \frac{u_k + i/2}{u_k - i/2}$

$$\Leftrightarrow u_k = \frac{1}{2} \cot \frac{p_k}{2}$$

Great reference: cond-mat/9809162.