Staudacher Lecture 2

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9:48 AM

Moral: Study the "Bethe Ansatz"

(ref: cond-mat/9809162)

Heisenberg spin chain - XXX model

$$\hat{H} = 4\sum_{l=1}^{L}\left(\frac{1}{4} - \vec{S}_l\cdot\vec{S}_{l+1}\right) \quad \text{w/ periodic boundary}$$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \qquad |\Psi\rangle \in (\mathbb{C}^2)^{\otimes L}$$

Q: How important is it that the scalar part of H would be just what it is?

Rewrite: $\hat{H} = \sum\left(1 - \vec{\sigma}_l\,\vec{\sigma}_{l+1}\right) =$

$$= \sum_{l=1}^{L}\left(1 - \sigma_l^3\sigma_{l+1}^3 - 2\sigma_l^+\sigma_{l+1}^- - 2\sigma_l^-\sigma_{l+1}^+\right)$$

$$= 2\sum_{l=1}^{L}\left(1 - P_{l,l+1}\right) = \sum \mathcal{X}_{l,l+1}$$

Permutation; i.e. $\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$

$$\begin{pmatrix}0&0&0&0\\0&2&-2&0\\0&-2&2&0\\0&0&0&0\end{pmatrix}$$

Due to conservation of total spin, the Hamiltonian decomposes into $\binom{L}{M}\times\binom{L}{M}$ blocks, $M = 0, \ldots, L$.

Shift operator: $\hat{U} = P_{12}P_{23}P \ldots P_{L-1,L}$

$$[\hat{U}, \hat{H}] = 0, \quad [\hat{U}, \vec{S}] = 0 \qquad \hat{U}^L = I$$

$\Rightarrow$ the eigenvalues $U = e^{\frac{2\pi i}{n}k}$.

Algebraic Solution: Introduce $u_k \in \mathbb{C}$ for each $\downarrow$ spin

Solve $$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

Then
$$E = 2\sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} \qquad U = \prod_{i=1}^{M} \frac{u_k + i/2}{u_k - i/2}$$

Proof 1 (coordinate Bethe ansatz, 1931)

write
$$|\psi\rangle = \sum_{l_1 < \ldots < l_M} \psi(l_1, \ldots, l_M) \underbrace{S_{l_1}^- S_{l_2}^- \cdots S_{l_M}^- |0\rangle}_{\parallel\ |\uparrow\uparrow\underset{l_1}{\downarrow}\uparrow\uparrow\underset{l_2}{\downarrow}\rangle}$$

$$\text{w/ } |0\rangle = |\uparrow\uparrow\uparrow\ldots\uparrow\rangle$$

The Ansatz:
$$\psi(l_1, \ldots, l_M) = \sum_{\tau \in S_M} A(\tau) e^{ip_{\tau_1} l_1 + \ldots + ip_{\tau_M} l_M}$$

Q: In the spirit of homology, can this image be written as a kernel?

So $|\downarrow\rangle$ is a "particle" called "magnon".

It works! w/
$$A(\tau) = \mathrm{sign}(\tau) \prod_{j<k} \left(e^{ip_k + ip_j} - 2e^{ip_k} + 1\right)$$

$$E = \sum_{k=1}^{M} 8 \sin^2 \frac{p_k}{2} \qquad U = \prod_{k=1}^{M} e^{ip_k}$$

On a finite lattice the p_k's are quantized:
$$e^{ip_k L} = \prod_{j=1, j\neq k}^{n} S(p_k, p_j)$$ ← The "S-

matrix=

$$S(p_k, p_j) = -\frac{e^{ip_k+ip_j} - 2e^{ip_k} + 1}{e^{ip_k+ip_j} - 2e^{ip_j} + 1}$$

The S-matrix here is "factorized", only 2-particle interactions exist.

Nice change of variables: $e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}$

$\Longleftrightarrow u_k = \frac{1}{2}\cot\frac{p_k}{2}$

Great reference: cond-mat/9809162.