

Relevant Sources

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From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004
Dror Bar-Natan, University of Toronto

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus, astrophysical lineups, one of the pillars of modern thought, are much older than the framed Gaussian linking number of two knots.

$(D, K)_n :=$ (The signed Stonehenge) (pairing of D and K)

$D =$ (diagram) $K =$ (diagram) \Rightarrow (Stonehenge diagram)

Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{n \rightarrow \infty} \sum_{D \in \mathcal{A}(K)} \frac{1}{2^n c(D)} (D, K)_n D \in \mathcal{A}(K)$$

$N := \#$ of stars $\mathcal{A}(K) :=$ oriented vertices
 $c := \#$ of chopsticks $\Rightarrow \text{Span} \left(\bigoplus_{AS} \mathbb{Z} \right) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
 $e := \#$ of edges of D $\Rightarrow \text{Span} \left(\bigoplus_{AS} \mathbb{Z} \right) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ & more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:
 A plane moves over an intersection point - Solution: Impose IHX.
 An intersection line cuts through the knot - Solution: Impose STU.
 The Gauss curve slides over a star - Solution: Multiply by a framing-dependent counter-term.

$I = H - X$ (see below) (similar argument) (not shown here)

The IHX Relation

It all is perturbative Chern-Simons-Witten theory:

$$\int DA \text{hol}_K(A) \exp \left[\frac{ik}{4\pi} \int U \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]$$

$\rightarrow \sum_{D: \text{Feynman diagrams}} W_D(D) \sum_{K: \text{Feynman diagrams}} \mathcal{E}(D) \rightarrow \sum_{D: \text{Feynman diagrams}} D \sum_{K: \text{Feynman diagrams}} \mathcal{E}(D)$

Shing-sha Chern James H Simons

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

The Gaussian linking number $lk(K_1, K_2) = \frac{1}{2} \sum_{\text{shaded chopsticks}} (\text{signs})$

Carl Friedrich Gauss Dehn Theorem

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More at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

From Stonehenge to Witten - Some Further Details

Oporto Meeting on Geometry, Topology and Physics, July 2004
Dror Bar-Natan, University of Toronto

We take the generating function of all stellar coincidences:

$$Z(K) := \lim_{n \rightarrow \infty} \sum_{D \in \mathcal{A}(K)} \frac{1}{2^n c(D)} (D, K)_n D \in \mathcal{A}(K)$$

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Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

$\int DA \text{hol}_K(A) \exp \left[\frac{ik}{4\pi} \int U \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]$

Related to Lie algebras

More precisely, let $\mathfrak{g} = (\mathcal{X}, \mathcal{Y})$ be a Lie algebra with an orthogonal basis, and let $R = (r_{ij})$ be a representation. Set $f_{ijk} := [r_i, r_j]_k$ $X_{ijk} := \sum_{\sigma \in S_3} r_{i\sigma(1)} r_{j\sigma(2)} r_{k\sigma(3)}$

Definition. \mathcal{Y} is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All nice polynomials (Coxeter, Jones, etc.) are of finite type.

Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. $Z(K)$ is a universal finite type invariant! (sketch to dance in many parties, you need many feet)

$W_{g,h} \oplus \mathbb{Z}$ is often interesting:

- $g = sl(2)$ The Jones polynomial
- $g = sl(N)$ The HOMFLYPT polynomial
- $g = so(N)$ The Kauffman polynomial

Planar algebras and the Yang-Baxter equation

Parentezed tangles, the pentagon and hexagon

Kauffman's bracket and the Jones polynomial

$\langle \bigcirc \rangle = 1$
 $\langle \bigcirc \rangle = (q + q^{-1}) \langle \bigcirc \rangle$
 $\langle \bigcirc \rangle = (q + q^{-1}) \langle \bigcirc \rangle$
 $\langle \bigcirc \rangle = (q + q^{-1}) \langle \bigcirc \rangle$

"God created the knots, all else in topology is the work of man."

Leopold Kronecker (modified)

More at <http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/>

Knotted Trivalent Graphs, Tetrahedra and Associators

HUIJ Topology and Geometry Seminar, November 16, 2000
Dror Bar-Natan

Goal: $Z(\text{knots}) \rightarrow (\text{chord diagrams}) \rightarrow \mathcal{A}$ so that

Module the relation(s):

Claim. With $\Phi := Z(\Delta)$, the above relation becomes equivalent to the Drinfeld's pentagon of the theory of quasi Hopf algebras.

Proof.

Extend to Knotted Trivalent Graphs (KTG's):

Need a new relation:

Easy, powerful moves:

Using moves, KTG is generated by ribbon twists and the tetrahedron

Further directions:

- Relations with perturbative Chern-Simons theory.
- Relations with the theory of 6j symbols
- Relations with the Turaev-Viro invariants
- Can this be used to prove the Witten asymptotics conjecture?
- Does this extend improve Drinfeld's theory of associators?

This handout is at <http://www.math.toronto.edu/~drorbn/Talks/HUIJ-001116>

More at <http://www.math.toronto.edu/~drorbn/Talks/HUIJ-001116/>

Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots
 Dror Bar-Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908

The Bigger Picture...
 Conventions statement: The Orbit Method
 Group-Algebra statement: Subject flow chart
 Unitary statement: Five Lie algebras
 Algebraic statement: Aho-Koschorik statement
 Diagrammatic statement: True
 Non-Theoretic statement: True

What are w-trivalent tangles?
 {knots} = PA / {R123, ...} / 0 legs
 {trivalent tangles} = PA / {R23, R4, ...} / 0 legs
 {trivalent w-tangles} = PA / generators relations / operations

W-trivalent w-tangles
 The w-generators: generators relations operations
 Virtual crossings: crossings
 Virtual crossing moves: crossings

Diagrammatic statement: Let $R = \exp^{\text{Lie}} \mathfrak{g}$ in $\mathcal{A}^n(\mathbb{R})$. There exist $\omega \in \mathcal{A}^n(\mathbb{R})$ and $V \in \mathcal{A}^n(\mathbb{R})$ so that

Diagrammatic to Algebraic: With (x_i) and (ψ^j) dual bases of \mathfrak{g} and \mathfrak{g}' with $\langle x_i, x_j \rangle = \delta_{ij}$, we have $\mathcal{A}^n \cong \mathcal{U}(\mathfrak{g})$ via

$$\omega \mapsto \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} \langle x_i, \psi^j \rangle \langle \psi^k, x_l \rangle \langle x_m, \psi^n \rangle \omega_{ijklm} \in \mathcal{U}(\mathfrak{g})$$

Unitary \Leftrightarrow Algebraic: The key is to interpret $\mathcal{U}(\mathfrak{g})$ as tangential differential operators on $\text{Func}(\mathfrak{g})$:
 • $\psi \in \mathfrak{g}'$ becomes a multiplication operator.
 • $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad}_x: (x\psi)(y) = \psi([x,y])$, and with $R = \exp^{\text{Lie}} \mathfrak{g}$ there exist $\omega \in \mathcal{S}(\mathfrak{g})$ and $V \in \mathcal{U}(\mathfrak{g})^{\text{inv}}$ so that
 (1) $V \psi^k = k! \langle \psi, x \rangle^k \omega$ (2) $V \psi^k = \sum_{j=0}^k \binom{k}{j} \langle \psi, x \rangle^j \psi^{k-j} \omega$
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Unitary \Rightarrow Group-Algebra: $\int \omega \psi^k = \sum_{j=0}^k \binom{k}{j} \langle \psi, x \rangle^j \int \omega \psi^{k-j}$
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Convolutions and Group Algebras (ignoring all Jacobians): If $\text{Func}(\mathfrak{g})$ is finite, A is an algebra, $\tau: G \rightarrow A$ is multiplicative then $\text{Func}(G, A) \cong (A, \tau)$ via $f \mapsto \sum (f \circ \tau)(g) \tau(g)$. For Lie (G, \mathfrak{g}) , $(g, \tau) \mapsto \sum_{g \in G} \tau(g) \psi^k \in \mathcal{S}(\mathfrak{g})$ and $\text{Func}(G, A) \xrightarrow{\tau} \mathcal{S}(\mathfrak{g})$. $\text{Func}(G, A) \xrightarrow{\tau} \mathcal{S}(\mathfrak{g})$. $\text{Func}(G, A) \xrightarrow{\tau} \mathcal{S}(\mathfrak{g})$.

Convolutions statement (Kashiwara-Vergne): Convolutions of invariant functions on a Lie group agree with convolutions of Lie algebras. More accurately, $\Phi^{-1}(\psi_1) \cdot \Phi^{-1}(\psi_2) = \Phi^{-1}(\psi_1 \cdot \psi_2)$ (inh. \mathcal{L}_i are "Lie algebra transformations"). Let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j, \psi: \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp: \mathfrak{g} \rightarrow G$, and let $\Phi: \text{Func}(G) \rightarrow \text{Func}(\mathfrak{g})$ be given by $\Phi(f)(x) = \int_{G/G_x} f(\exp(x)g) dg$. Then if $f, g \in \text{Func}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$.

Example: Pure Braids: PB_n is generated by s_{ij} "strand i goes around strand j once", modulo "Reidemeister moves". $A_n = \mathfrak{g} PB_n$ is generated by $t_{ij} = s_{ij} - 1$, modulo the 4F relations $t_{ij} t_{jk} - t_{jk} t_{ij} = 0$ (and some lesser ones too). Much happens in A_n , including the Drinfeld theory of associators.

Our case: K is knot theory or topology; $\mathfrak{g}K$ is finite combinatorics; bounded-complexity diagrams modulo simple relations.

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/>

Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots, Page 2
 Knot-theoretic statement: There exists a homomorphism $\text{exp}^{\text{Lie}} \mathfrak{g} \rightarrow \mathcal{A}^n(\mathbb{R})$. For $n=3$ we have $\text{exp}^{\text{Lie}} \mathfrak{g} \cong \mathcal{A}^3(\mathbb{R})$. For $n=4$ we have $\text{exp}^{\text{Lie}} \mathfrak{g} \cong \mathcal{A}^4(\mathbb{R})$. For $n=5$ we have $\text{exp}^{\text{Lie}} \mathfrak{g} \cong \mathcal{A}^5(\mathbb{R})$.

Diagrammatic statement: Let $R = \exp^{\text{Lie}} \mathfrak{g}$ in $\mathcal{A}^n(\mathbb{R})$. There exist $\omega \in \mathcal{A}^n(\mathbb{R})$ and $V \in \mathcal{A}^n(\mathbb{R})$ so that

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W-Knots from Z to A: Dror Bar-Natan, Luny, April 2010. Abstract: I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant Z of w-knots. In order to study Z we will introduce the "Euler Operator" and the "Infinitesimal Alexander Module", at the end finding a simple determinant formula for Z. With this Z we find a formula that computes the Alexander polynomial except I don't have a proof yet.

The Bracket-Rise Theorem: \mathcal{A}^n is isomorphic to \mathcal{A}^n (2 in 1 out vertices) and \mathcal{A}^n relations.

The Alexander Theorem: $T_{ij} = \text{low}(\#ij)$, $d_i = \text{dir}(\#ij)$, $S = \text{diag}(a_{ij})$, $A = \det(V - T(S - S))$. $X^2 = \text{diag}(a_{ij})$, $Y = \text{diag}(a_{ij})$, $Z = N \exp_{\mathbb{R}}(-w \log_{\mathbb{Q}(i)} A(e^{\psi}))$ mod $u_1 u_2 = u_2 u_1$, $Z = N \cdot A^{-1}(e^{\psi})$.

Conjecture: For ψ -knots, A is the Alexander polynomial. Theorem: With $w: \mathbb{R}^3 \rightarrow u_3$ (the k-wheel), $Z = N \exp_{\mathbb{R}}(-w \log_{\mathbb{Q}(i)} A(e^{\psi}))$ mod $u_1 u_2 = u_2 u_1$, $Z = N \cdot A^{-1}(e^{\psi})$.

Proof Sketch: Let E be the Euler operator, "multiply anything by its degree", $f = z^d$ in $\mathbb{Q}[z]$ to $E z^d = z^{d+1}$ and we need to show that $Z \cdot E Z = N \cdot \text{tr}((I - B) \cdot T(S - S)) u_1$ with $B = T(e^{-\text{ad} \cdot Z})$. Note that $ae^{\psi} \cdot e^{\psi} = (1 - e^{-\text{ad} \cdot \psi})(e^{\psi})$ implies $E Z = N \cdot \text{tr}((I - B) \cdot T(S - S)) u_1$.

So What? Habiro-Ekman did this already, but not quite. (See Finite Type Invariants of Ribbon 2-Knots, Z. Top. and its Appl. 111 (2003)). New (?) formula for Alexander, see (?) "Infinitesimal Alexander Module". Related to Lempert's arXiv:1001.4474. An "ultimate Alexander invariant" book, compose well, behave under calling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. Tip of the Alexander-Teressian-Kashiwara-Vergne isobring (AT). The Kashiwara-Vergne conjecture and Drinfeld's associator, arXiv:0802.0200. Tip of the \mathbb{Z} -knots isobring. May lead to other polynomial-time polynomial invariants. "A polynomial" worth a thousand exponentials. Also see <http://www.math.toronto.edu/~drorbn/papers/WK0/>.

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/Luny-1004/>

18 Conjectures
 Dror Bar-Natan, Chicago, September 2010. <http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009>

Abstract: I will state 18 = 3 x 3 x 2 "fundamental" conjectures on finite type invariants of various classes of virtual knots. This done, I will state a few further conjectures about these conjectures and ask a few questions about how these 18 conjectures may or may not interact.

Following "Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots", by B.N. Habiro, Leung, and Rouquier, <http://www.math.toronto.edu/~drorbn/papers/v-Dims/>

invariant/algebra	virtual	inv	non-inv	inv	non-inv
standard	mod R1	0, 1, 2, 7, 9	0, 2, 7, 9, 249	0, 1, 6, 34	
R2b, R2c, R3b	mod R1	1, 4, 7, 29	2, 4, 15, 67, 365	1, 15, 4, 34	
head-like	mod R1	0, 1, 5, 17	0, 7, 42, 249	0, 1, 6, 34	
R2b, R3b	mod R1	0, 1, 5, 17	0, 7, 42, 249	0, 1, 6, 34	
R2b, R3b	mod R1	0, 1, 5, 17	0, 7, 42, 249	0, 1, 6, 34	
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Comments: These 18 are the "standard" virtual conjectures. Comments: These 18 are the "standard" virtual conjectures. Comments: These 18 are the "standard" virtual conjectures.

Definitions: $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$. $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$.

The Polya's Technique: $eK = CA_2 \langle \mathcal{A}^n \rangle / \mathcal{R}^n = \{KT, \dots\}$ the case

Warning! $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$. $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$, $\mathcal{A}^n = \mathcal{A}^n$.

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