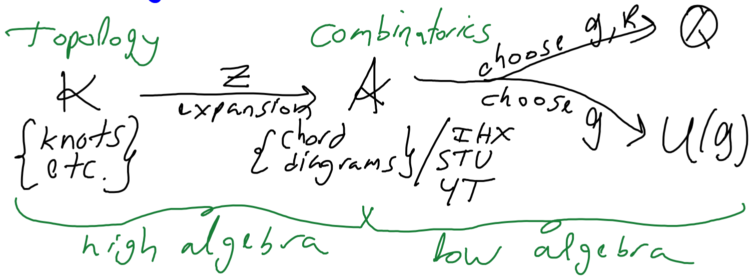
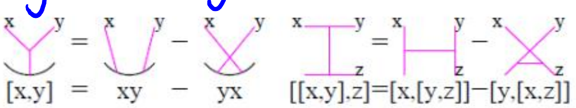
Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.



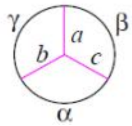
Very low algebra.



More precisely, let $\mathfrak g = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R=\langle v_\alpha\rangle$ be a representation. Set

$$f_{abc} := \langle [a,b],c\rangle \qquad X_a v_\beta = \sum_{\gamma} r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak g,R}:$ 

$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc}\, r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{\mathfrak g,R}(D)$ when $\mathfrak g=\mathfrak{gl}_n$, $R=\mathbb R^n$? Is it related to the Conway polynomial?

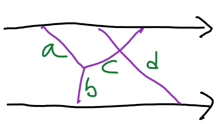
Universal Representation Theory.

Inspired by $\rho([x,y])=\rho(x)\rho(y)-\rho(y)\rho(x)$, set $U(\mathfrak g) = \langle \text{words in } \mathfrak g\rangle/[x,y]=xy-yx$

* Every rep of $\mathfrak g$ extends to $U(\mathfrak g)$.
* $\exists\, \Delta: U(\mathfrak g)\to U(\mathfrak g)^{\otimes 2}$ by "word splitting", as must be for $R_1\otimes R_2$.

Exercise. With $\mathfrak g = \langle x,y\rangle/[x,y]=x$, determine $U(\mathfrak g)$. Guess a generalization.

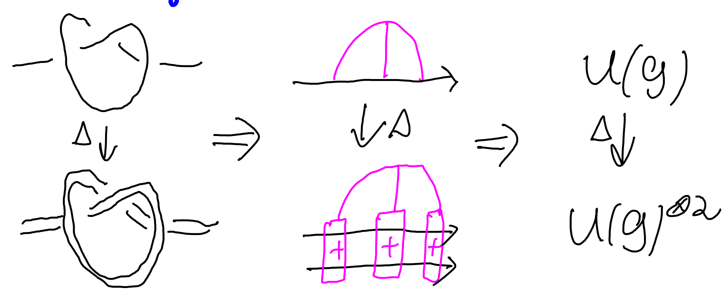
Low algebra. $\mathcal A(\uparrow\uparrow)\to U(\mathfrak g)^{\otimes 2}$ via



$$\longrightarrow \sum_{a-d} f_{abc}\begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

& likewise, $\mathcal A(\uparrow_n)\to U(\mathfrak g)^{\otimes n}$ $\Rightarrow$ $\mathcal A(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

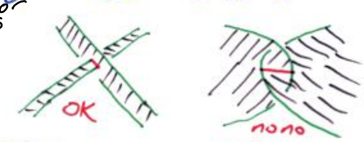


A "Homomorphic Expansion" $Z:\mathcal K\to\mathcal A$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal K$ is finitely presented, finding Z is **High Algebra**.

$\{\text{Ribbon knots}\} \overset{\uparrow\text{ more or less}}{=} \{u\gamma : \gamma\in\mathcal K(\bigcirc\!-\!\bigcirc),\ d\gamma = \bigcirc\bigcirc\}$

Where Ribbon means



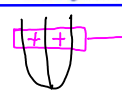
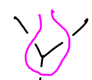
and

Algebraic knot Theory:

$$\begin{tikzcd} & \mathcal K(\bigcirc\bigcirc) \arrow[r,"Z"] & \mathcal A(\bigcirc\bigcirc)\ni\bigcirc\bigcirc \\ \mathcal K(\bigcirc\!-\!\bigcirc) \arrow[ur,"d"] \arrow[rr,"Z"] \arrow[dr,"u"'] & & \mathcal A(\bigcirc\!-\!\bigcirc) \arrow[u,"d"'] \arrow[d,"u"] \\ & \mathcal K(\ominus) \arrow[r,"Z"] & \mathcal A(\bigcirc) \end{tikzcd}$$

So

$$Z(\{\text{Ribbon knots}\})\subset\{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\}\subset\mathcal A(\bigcirc\!-\!\bigcirc)$$

VI  $=0$, follows from  $=$ 

An Associator:
Quantum Algebra's "root object"

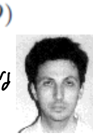
$$(AB)C \xrightarrow{\Phi\in U(\mathfrak g)^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{tikzcd} ((AB)C)D \arrow[rr,"(\Delta11)\Phi"] \arrow[d,"\Phi1"'] & & (AB)(CD) \arrow[d,"(11\Delta)\Phi"] \\ (A(BC))D \arrow[dr,"(1\Delta1)\Phi"'] & & A(B(CD)) \\ & A((BC)D) \arrow[ur,"1\Phi"'] & \end{tikzcd}$$

$$\Phi1\cdot(1\Delta1)\Phi\cdot1\Phi=(\Delta11)\Phi\cdot(11\Delta)\Phi$$

V.G. Drinfel'd



The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv:1103.1896