Lecture 3 Handout

The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

$$C:\{\text{knot diagrams}\}/(\text{Reidemeister moves}) \to \mathbb{Z}[z]$$

Example. The Conway polynomial is given by

$$C(\nearrow\!\!\!\!\nwarrow) - C(\nwarrow\!\!\!\!\nearrow) = zC(\,)(\,)$$

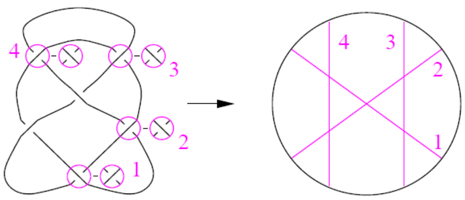
and

$$C(\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{k}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

INTERLINK

Definition. Any $V:\{\text{knots}\}\to$ Abelian Group A can be extended to "knots w/ double points" using $V(\times\!\!\!\!\bullet) = V(\nearrow\!\!\!\!\nwarrow) - V(\nwarrow\!\!\!\!\nearrow)$. (Think "differentiation")



Definition. V is of type m if always

$$V(\underbrace{\times\!\!\!\!\bullet\ \times\!\!\!\!\bullet\ \cdots\ \times\!\!\!\!\bullet}_{m+1}) = 0 \qquad (\text{think "polynomial"})$$

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K)=\sum_{m=0}^{\infty} V_m(K)z^m$ then V_m is of type m.

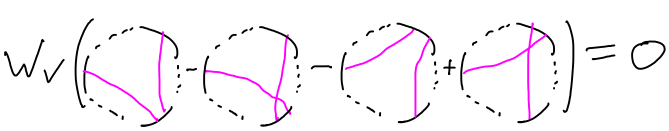
Proof. $C(\times\!\!\!\!\bullet) = C(\nearrow\!\!\!\!\nwarrow) - C(\nwarrow\!\!\!\!\nearrow) = zC(\,)(\,)$ $\square$

Let V be of type m; then $V^{(m)}$ is constant:

$$V(\underbrace{\times\!\!\!\!\bullet \cdots \times\!\!\!\!\bullet}_{m}\nearrow\!\!\!\!\nwarrow) = V(\times\!\!\!\!\bullet \cdots \times\!\!\!\!\bullet\nwarrow\!\!\!\!\nearrow)$$

So $W_V := V^{(m)} = V\big|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V:\{\text{chord diagrams}\}\to A$

Claim. W_V satisfies the 4T relation:



Proof. $V(\,\cdot\,) = V(\,\cdot\,)$ $\square$

$\times\cdot\times\ m-2$ $\qquad$ $\times\cdots\times\ m-2$

Exercise for Lecture 2. Use $\int_{\mathbb{R}} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-\|x\|^2/2}d^nx$ in two different ways and hence to deduce the volume of S^{n-1}, the $(n-1)$-dimensional sphere.

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.

2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

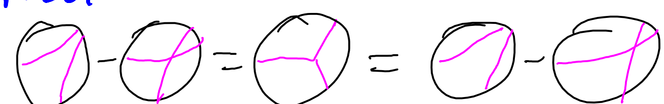
Theorem. **(The Fundamental Theorem)** Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W=W_V$


M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Theorem. $\mathcal{A}^{\text{today}} \cong \mathcal{A}^{\text{Monday}}$

Proof



Proposition. The fundamental thm holds iff there exists an expansion: $Z:\mathcal{K}\to\hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.

Proof.

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{Z} & \mathcal{A} \\ & \searrow^{V} & \downarrow W \\ & & \mathbb{Q} \end{array} \qquad \square$$

Also see my old paper, "On the Vassiliev knot invariants" (google will find...)

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In[1]:= << KnotTheory`

  Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
  Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
      Import[
       "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
        <> ToString@#[[1]] <>
        "." <> ToString@#[[2]] <> "_240.gif"],
      Conway[#][z]
     }, Center
    ] & /@ AllKnots[{0, 7}]

  KnotTheory::loading : Loading precomputed data in PD4Knots`.
```


J.H. Conway