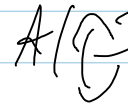
Ishii: A quandle cocycle invariant for handlebody links

July-22-11
10:19 AM

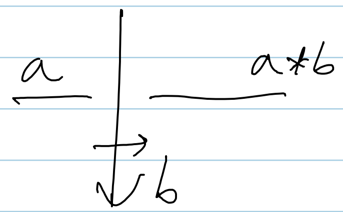
__Def__ $(X, *)$: A quandle:

1. $a * a = a$
2. $\cdot * a : X \to X$ is a bijection
3. $(a*b)*c = (a*c)*(b*c)$

__Def__ D: a diag of an oriented link

$A(D) := \{\text{arcs of } D\}$ $\quad$ ($A($  $) = 3$ arcs)

$C : A(D) \to X$ is an X-coloring if



at each crossing

$\mathrm{Col}_X(D) = \{X\text{-colorings of } D\}$

__Prop__ If $D \xleftrightarrow{\text{R-moves}} D'$ then $\exists$ a bijection

$$\mathrm{Col}_X(D) \to \mathrm{Col}_X(D')$$

Quandle (co)homology:

$$B_n(X) := \mathbb{Z}[X^n]$$

$\partial_n : B_n(X) \to B_{n-1}(X)$ by

$$(q_1, \dots, q_n) \mapsto \sum_{i=1}^{n} (-1)^i (q_1 \wedge q_i, \dots, q_{i-1} \wedge q_i, q_{i+1}, \dots, q_n)$$

$$\sum_{i=1}(-1)^i(q_1,\ldots \hat{q_i},\ldots q_n)$$

--- define H_n^R "Rack homology"

$D_n(X)\subset B_n(X)$: The subgroup of $B_n(X)$ generated by sequences w/ a "doubled" element: $(q_1 \ldots q_{i-1}, q, q, q_{i+2}, \ldots)$

Claim $D_n(X)$ is a subcomplex.

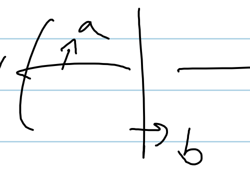
Def $C_n(X) := B_n(X)/D_n(X)$

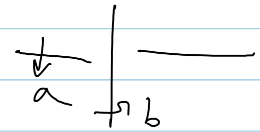
$\Rightarrow$ define $H_n^Q(X)$ "quandle homology"

Def For $C\in Col_X(D)$, define

$$W(D;C)=\sum_{x\in \text{xings of } D} w(x,C)$$

where

$w($$) := (a,b)\in C_2(X)$

$w($$) := -(a,b)\in C_2(X)$

Lemma $\partial_2(W(D,C))=0$

Prop If D & D' are related by one of R_1, R_2 or R_3, then

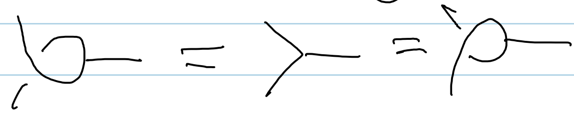
$\lfloor W(D,C)\rfloor = \lfloor W(D', C_{D,D'})\rfloor$ in $H_2(X)$

$\Longrightarrow$ $H(D)=\{[W(D,C)] : C\in \mathrm{Col}_X(D)\}$
$\Phi_\theta(D) := \{\theta(W(D,C)) : C \in \mathrm{Col}_X(D)\}$ } are link invariants
for $\theta\in H^2(X)$

A quandle colouring for handlebody-links:

Def A handlebody-link is an embedding of a disjoint union of handlbodies into $\mathbb{R}^3$, up to isotopy.

Thm 2 diagrams of handlebody links are equivalent iff they differ by R1–R3, R4, I $\Leftrightarrow$ H,



Def Let G be a group.

$(X, \{*^g : g\in G\})$ is a G-family of quandles if

* $a *^g a = a$
* $a *^e b = a \qquad a*^{gh} b = (a*^g b)*^h b$
* $(a*^g b)*^h c = (a*^h c)*^{h^{-1}gh}(b*^h c)$

Is there an infinitesimal "Leibniz" analog of that?

Prop 1. $(X,*^g)$ is a quandle for every g.

2. $(a,g)*(b,h) := (a*^h b,\ h^{-1}gh)$

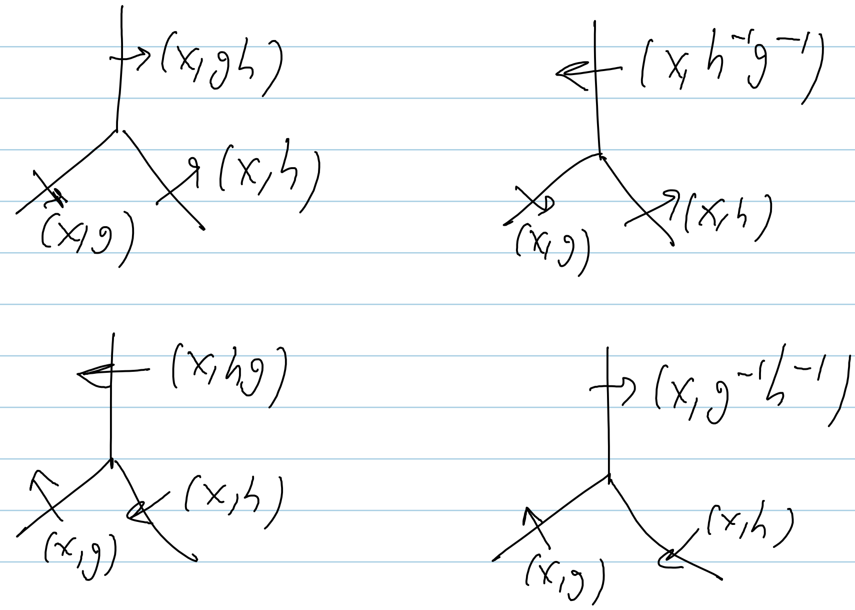
makes $Q = X \times G$ into a quandle.

"the associated quandle of X"

<u>Def</u> An X-colouring of a handlebody diagram D is

1. The usual Q-condition near Xings.

Also,



<u>Thm</u> This is "naturally" invariant under R1–R6.

<u>Claim</u> There is a revised definition for $D_n(X)$ for this case; the rest of the story also goes through.

Examples: 1. If X is a quandle, let

$$a *^n b := ((a * b) * b) * b \ldots \text{ n times}$$

then $(X, *^n)$ is a $\mathbb{Z}$-family of quandles.

2. If G is a group & R a ring, X is a right $R[G]$-module, then

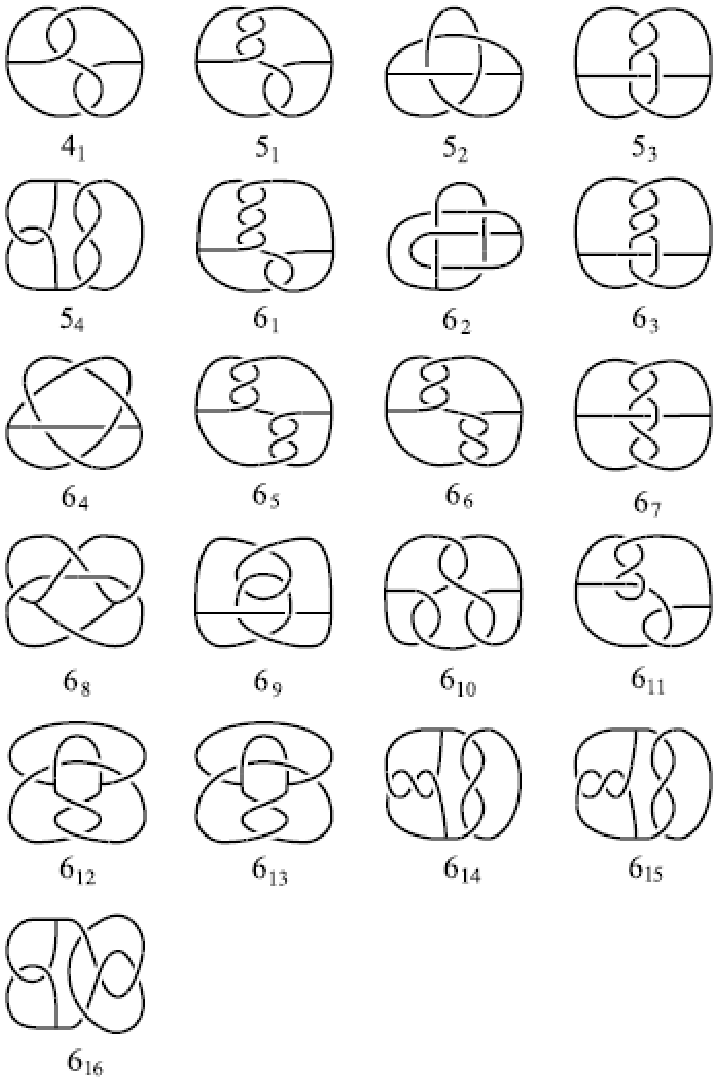
$$a *^g b := a \cdot g + b(1-g)$$

is a G-family of quandles.

See a tabulation at

http://www.math.tsukuba.ac.jp/~aishii/files/paper019.pdf

Includes -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