Pensieve Header: Computations in the universal enveloping algebra of gl(*), with Emily Cliff and Iva Halacheva.

```
Unprotect[NonCommutativeMultiply];

b[E[s_, ij__], E[t_, kl__]] /; s ≠ t := 0;
b[E[s_, i_, j_], E[s_, k_, l_]] :=
  KroneckerDelta[j, k] W[E[s, i, l]] - KroneckerDelta[i, l] W[E[s, k, j]];
b[_, _a] = 0;
b[_a, _] = 0;
b[_, _q] = 0;
b[_q, _] = 0;
b[x_, y_] := x ** y - y ** x;

PBWReduce[W[]] = RW[];
PBWReduce[W[e_, more___]] := W[e] ** PBWReduce[W[more]];
PBWReduce[expr_] := Expand[expr /. w_W :> PBWReduce[w]];
_ ** 0 = 0;
x_ ** (c_?NumberQ * y_) := c (x ** y);
x_ ** y_Plus := (x ** #) & /@ y;
0 ** _ = 0;
(c_?NumberQ * y_) ** x_ := c (y ** x);
x_Plus ** y_ := (# ** y) & /@ x;
W[w1___] ** W[w2___] := W[w1, w2];
W[e_] ** W[] := RW[e];
W[e_] ** RW[] := RW[e];

W[e_] ** RW[f_, more___] /; OrderedQ[{e, f}] := RW[e, f, more];
W[e_] ** RW[f_, more___] /; !OrderedQ[{e, f}] := Plus[
   W[f] ** (W[e] ** RW[more]),
   b[e, f] ** RW[more]
  ];

Sh[0, l_] := {{l, {}, Range[l]}};
Sh[k_, 0] := {{k, Range[k], {}}};
Sh[k_, l_] := Sh[k, l] = Join[
    Replace[Sh[k - 1, l] + 1, {n_, {is___}, {js___}} :> {n, {1, is}, {js}}, {1}],
    Replace[Sh[k, l - 1] + 1, {n_, {is___}, {js___}} :> {n, {is}, {1, js}}, {1}],
    Replace[Sh[k - 1, l - 1] + 1, {n_, {is___}, {js___}} :> {n, {1, is}, {1, js}}, {1}]
   ];

ReIndex[E[s_, i_, j_], is_List] := E[s, is[[i]], is[[j]]];
ReIndex[a[i_, j_], is_List] := a[is[[i]], is[[j]]];
ReIndex[q[i_, j_], is_List] := q[is[[i]], is[[j]]];

(S[k_] w1_W) ** (S[l_] w2_W) := Total[
   (
     S[#[[1]]] NonCommutativeMultiply[
       Replace[w1, e_ :> ReIndex[e, #[[2]]], {1}],
       Replace[w2, e_ :> ReIndex[e, #[[3]]], {1}]
      ]
    ) & /@ Sh[k, l]
```

```
  ];

Act[ss_List, expr_] := expr /. E[s_, ij__] :> E[ss[[s]], ij];

r = Plus[
   1 / 2 S[1] W[E[1, 1, 1], E[2, 1, 1]],
   S[2] W[a[1, 2], E[1, 1, 1], E[2, 2, 2]],
   -S[2] W[a[1, 2], E[2, 1, 1], E[1, 2, 2]],
   S[2] W[E[1, 2, 1], E[2, 1, 2]]
  ];
CYBE[r_] := Plus[
   b[Act[{1, 2}, r], Act[{1, 3}, r]],
   b[Act[{1, 2}, r], Act[{2, 3}, r]],
   b[Act[{1, 3}, r], Act[{2, 3}, r]]
  ];

TStar[r_, diag_Diag] := TStar[r, diag] = PBWReduce[
    PBWReduce[
      NonCommutativeMultiply @@ (diag /. ar[s1_, s2_] :> Act[{s1, s2}, r])
     ] /. {
      E[s_, ij__] :> E[1, ij],
      RW → W
     }
   ];
TStar[r_, expr_] := Expand[expr /. diag_Diag :> TStar[r, diag]];
TStar[expr_] := TStar[r, expr];

PBWReduce[S[2] W[e[1, 2, 1], e[2, 1, 2], e[1, 2, 1], e[3, 1, 2]]]

RW[e[1, 2, 1], e[1, 2, 1], e[2, 1, 2], e[3, 1, 2]] S[2]

CYBE[r] // PBWReduce

0

PBWReduce[W[E[2, 6, 7], E[1, 4, 2], E[1, 2, 3]]]

RW[e[1, 4, 3], e[2, 6, 7]] + RW[e[1, 2, 3], e[1, 4, 2], e[2, 6, 7]]

PBWReduce[W[E[1, 2, 3], E[1, 2, 4]]]

RW[e[1, 2, 3], e[1, 2, 4]]

PBWReduce[W[E[2, 1, 3], E[1, 3, 4]]]

RW[e[1, 3, 4], e[2, 1, 3]]

Sh[2, 3]

{{5, {1, 2}, {3, 4, 5}}, {5, {1, 3}, {2, 4, 5}},
 {5, {1, 4}, {2, 3, 5}}, {5, {1, 5}, {2, 3, 4}},
 {4, {1, 4}, {2, 3, 4}}, {4, {1, 3}, {2, 3, 4}}, {4, {1, 2}, {2, 3, 4}},
 {5, {2, 3}, {1, 4, 5}}, {5, {2, 4}, {1, 3, 5}}, {5, {2, 5}, {1, 3, 4}},
 {4, {2, 4}, {1, 3, 4}}, {4, {2, 3}, {1, 3, 4}}, {5, {3, 4}, {1, 2, 5}},
 {5, {3, 5}, {1, 2, 4}}, {4, {3, 4}, {1, 2, 4}}, {5, {4, 5}, {1, 2, 3}},
 {4, {3, 4}, {1, 2, 3}}, {4, {2, 3}, {1, 2, 4}}, {4, {2, 4}, {1, 2, 3}},
 {3, {2, 3}, {1, 2, 3}}, {4, {1, 2}, {1, 3, 4}}, {4, {1, 3}, {1, 2, 4}},
 {4, {1, 4}, {1, 2, 3}}, {3, {1, 3}, {1, 2, 3}}, {3, {1, 2}, {1, 2, 3}}}
```

```
r
```

```
1/2 S[1] W[𝕖[1, 1, 1], 𝕖[2, 1, 1]] + S[2] W[𝕖[1, 2, 1], 𝕖[2, 1, 2]] +
 S[2] W[a[1, 2], 𝕖[1, 1, 1], 𝕖[2, 2, 2]] - S[2] W[a[1, 2], 𝕖[2, 1, 1], 𝕖[1, 2, 2]]
```

```
X = S[2] W[q[1, 2], E[1, 1, 2]]
```

```
S[2] W[q[1, 2], 𝕖[1, 1, 2]]
```

```
PBWReduce[b[Act[{1}, X], r] + b[Act[{2}, X], r]]
```

```
1/2 RW[𝕖[1, 1, 1], 𝕖[2, 1, 2], q[1, 2]] S[2] - 1/2 RW[𝕖[1, 1, 2], 𝕖[2, 1, 1], q[1, 2]] S[2] +
 1/2 RW[𝕖[1, 1, 2], 𝕖[2, 2, 2], q[1, 2]] S[2] - 1/2 RW[𝕖[1, 2, 2], 𝕖[2, 1, 2], q[1, 2]] S[2] +
 RW[a[1, 2], 𝕖[1, 1, 1], 𝕖[2, 1, 2], q[1, 2]] S[2] -
 RW[a[1, 2], 𝕖[1, 1, 2], 𝕖[2, 1, 1], q[1, 2]] S[2] -
 RW[a[1, 2], 𝕖[1, 1, 2], 𝕖[2, 2, 2], q[1, 2]] S[2] +
 RW[a[1, 2], 𝕖[1, 2, 2], 𝕖[2, 1, 2], q[1, 2]] S[2] + RW[𝕖[1, 1, 2], 𝕖[2, 2, 3], q[1, 3]] S[3] -
 RW[𝕖[1, 2, 3], 𝕖[2, 1, 2], q[1, 3]] S[3] - RW[a[1, 2], 𝕖[1, 1, 1], 𝕖[2, 2, 3], q[2, 3]] S[3] -
 RW[a[1, 2], 𝕖[1, 1, 3], 𝕖[2, 2, 2], q[1, 3]] S[3] +
 RW[a[1, 2], 𝕖[1, 2, 2], 𝕖[2, 1, 3], q[1, 3]] S[3] +
 RW[a[1, 2], 𝕖[1, 2, 3], 𝕖[2, 1, 1], q[2, 3]] S[3] +
 RW[a[1, 3], 𝕖[1, 1, 1], 𝕖[2, 2, 3], q[2, 3]] S[3] -
 RW[a[1, 3], 𝕖[1, 1, 2], 𝕖[2, 3, 3], q[1, 2]] S[3] -
 RW[a[1, 3], 𝕖[1, 2, 3], 𝕖[2, 1, 1], q[2, 3]] S[3] +
 RW[a[1, 3], 𝕖[1, 3, 3], 𝕖[2, 1, 2], q[1, 2]] S[3] +
 RW[a[2, 3], 𝕖[1, 1, 2], 𝕖[2, 3, 3], q[1, 2]] S[3] -
 RW[a[2, 3], 𝕖[1, 1, 3], 𝕖[2, 2, 2], q[1, 3]] S[3] +
 RW[a[2, 3], 𝕖[1, 2, 2], 𝕖[2, 1, 3], q[1, 3]] S[3] -
 RW[a[2, 3], 𝕖[1, 3, 3], 𝕖[2, 1, 2], q[1, 2]] S[3]
```

```
Rels[2]
```

```
{Diag[ar[1, 2], ar[3, 4]] + Diag[ar[1, 3], ar[2, 4]] - 2 Diag[ar[1, 4], ar[2, 3]],
 Diag[ar[1, 2], ar[4, 3]] - Diag[ar[1, 3], ar[2, 4]] +
  Diag[ar[1, 4], ar[2, 3]] - Diag[ar[1, 4], ar[3, 2]], -Diag[ar[1, 3], ar[2, 4]] +
  Diag[ar[1, 4], ar[2, 3]] - Diag[ar[1, 4], ar[3, 2]] + Diag[ar[2, 1], ar[3, 4]],
 -Diag[ar[2, 1], ar[3, 4]] + Diag[ar[2, 3], ar[4, 1]] + Diag[ar[3, 1], ar[4, 2]] -
  Diag[ar[3, 2], ar[4, 1]], -Diag[ar[1, 2], ar[4, 3]] +
  Diag[ar[2, 3], ar[4, 1]] + Diag[ar[3, 1], ar[4, 2]] - Diag[ar[3, 2], ar[4, 1]],
 -Diag[ar[2, 1], ar[4, 3]] - Diag[ar[3, 1], ar[4, 2]] + 2 Diag[ar[3, 2], ar[4, 1]]}
```

```
TStar[Rels[2]]
```

```
{0, 0, 0, 0, 0, 0}
```

```
Rels[3]
```

A very large output was generated. Here is a sample of it:

```
{Diag[ar[1, 2], ar[3, 4], ar[5, 6]] +
  Diag[ar[1, 2], ar[3, 5], ar[4, 6]] - 2 Diag[ar[1, 2], ar[3, 6], ar[4, 5]],
 Diag[ar[1, 2], ar[3, 4], ar[6, 5]] - Diag[ar[1, 2], ar[3, 5], ar[4, 6]] +
  Diag[ar[1, 2], ar[3, 6], ar[4, 5]] - Diag[ar[1, 2], ar[3, 6], ar[5, 4]], <<117>>,
 -Diag[ar[3, 2], ar[4, 1], ar[6, 5]] + Diag[ar[3, 2], ar[5, 4], ar[6, 1]] -
  Diag[ar[4, 3], ar[5, 1], ar[6, 2]] + Diag[ar[4, 3], ar[5, 2], ar[6, 1]]}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
TStar[Diag[ar[1, 3], ar[4, 2]]] /. RW[_E, ___] :> 0
```

```
RW[a[1, 2], e[1, 1, 1], e[1, 1, 1]] S[2] - RW[a[1, 2], e[1, 2, 2], e[1, 2, 2]] S[2] +
 RW[a[1, 2], e[1, 1, 1], e[1, 3, 3]] S[3] - RW[a[1, 2], e[1, 2, 2], e[1, 3, 3]] S[3] +
 RW[a[1, 3], e[1, 1, 1], e[1, 2, 2]] S[3] - RW[a[1, 3], e[1, 2, 2], e[1, 3, 3]] S[3] +
 RW[a[2, 3], e[1, 1, 1], e[1, 2, 2]] S[3] - RW[a[2, 3], e[1, 1, 1], e[1, 3, 3]] S[3]
```

```
Print[TStar[#]] & /@ Rels[3];
```

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Length[**BasisAArrow**[**3**]]

```
27
```

```
im = TStar[BasisAArrow[3]]
```

A very large output was generated. Here is a sample of it:

```
{1/8 RW[e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1]] S[1] -
  1/2 RW[e[1, 1, 1], e[1, 1, 2], e[1, 2, 1]] S[2] + <<546>> +
  6 RW[e[1, 1, 6], e[1, 2, 5], e[1, 3, 4], e[1, 4, 3], e[1, 5, 2], e[1, 6, 1]] S[6],
 <<25>>, <<1>>}
```

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```
TargetBasis = Union[Cases[
   TStar[DomainBasis = BasisAArrow[3]] /. c_?NumberQ * rw_RW * n_S :> n * rw,
   rw_RW * n_S,
   Infinity
  ]]
```

A very large output was generated. Here is a sample of it:

```
{RW[e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1]] S[1],
 RW[e[1, 1, 1]] S[2], <<1961>>,
 RW[e[1, 1, 6], e[1, 2, 5], e[1, 3, 4], e[1, 4, 3], e[1, 5, 2], e[1, 6, 1]] S[6]}
```

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```
rule = Thread[TargetBasis → IdentityMatrix[Length[TargetBasis]]]
```

A very large output was generated. Here is a sample of it:

```
{RW[e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1], e[1, 1, 1]] S[1] →
  {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, <<1910>>,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, <<1963>>}
```

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```
mat = (im /. rule)
```

A very large output was generated. Here is a sample of it:

```
{<<1>>}
```

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```
Dimensions[mat]
```

```
{27, 1964}
```

```
mat[[1]]
```

$$\left\{\frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, -1, 0, -1, 0, 0, 0, \frac{5}{4}, 0, -\frac{3}{2}, 0, 2, \frac{5}{4},\right.$$

0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $-\frac{3}{2}$, 0, $\frac{3}{2}$, 0, −3, $-\frac{3}{2}$, 0, 3, $\frac{3}{2}$, 0, 1, 2, 1, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{2}$,
$\frac{3}{2}$, $\frac{3}{8}$, 1, $\frac{3}{2}$, $\frac{3}{4}$, 0, 0, 0, 0, 0, 0, $-\frac{1}{2}$, 0, 0, 0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 2, −2, −1, 0, 0, 0, 0, 0,
0, 0, 0, 2, 0, 0, 1, 0, 0, 0, 2, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −2, 0, 0, −1, 0, 0,
−1, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, −2, −3, −2, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, −1, 0, 0, $\frac{1}{2}$, 0, 0, 0, $-\frac{3}{2}$, $\frac{3}{2}$, 0, 0, 0, 0, 0, 0, 0, 0, 0, −1, 0, 0, 1, $\frac{1}{2}$,
−3, $\frac{3}{2}$, −2, −3, $-\frac{3}{2}$, $-\frac{1}{2}$, 0, −1, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, −2, 2, 0, 0, 0, 0, 0, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 1, −1, 0, 0, 0, 0, 0, 0, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, −1, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
$\frac{3}{2}$, $-\frac{3}{2}$, 0, 0, 0, $-\frac{3}{2}$, $\frac{3}{2}$, 0, 0, −6, 0, −3, $-\frac{3}{2}$, 0, $-\frac{3}{2}$, −3, 0, 0, 0, 0, 0, 0, 0, 0, 0,
3, −3, 3, 3, 3, 0, 0, $\frac{3}{2}$, 3, 0, $\frac{3}{2}$, 3, $\frac{3}{2}$, −3, 0, $-\frac{3}{2}$, −3, $-\frac{3}{2}$, $\frac{3}{2}$, −3, 6, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1, 1, −2, 2, 2, 2, 2, −2, −2, −2, −2, 2, 1, 1, 2, 2, 2, −2, 2, 1, 1,
$\frac{3}{4}$, 3, 3, $\frac{3}{2}$, $\frac{3}{2}$, 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{2}$, $\frac{3}{2}$, 3, 3, $\frac{3}{2}$, 3, 3, 6, 3, 3, $\frac{3}{2}$, 3, 3, $\frac{3}{4}$, $\frac{3}{2}$, 3, $\frac{3}{4}$, 3, $\frac{3}{2}$,
3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, −2, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, −2, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −2, 0, 0, 0, 0, −2, 0, 0, 2, 0, 0,
0, 0, 0, 1, −1, 0, 0, −4, 0, 0, 0, −1, 0, 0, 0, 0, 0, −2, −4, 0, 0, −2, −1, 0, 0, 0, 1,
0, 0, −1, 0, 0, 0, −4, −2, −4, 0, 0, −1, 0, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, −1, 0, 0, 0, 0, 0, 0, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −2, 2, 0, 0, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, −1, 0, 0, −1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, −2, 0, −2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 2, −2, 0, 0, 1, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
−1, 1, 1, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $\frac{3}{2}$, $-\frac{3}{2}$,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −3, 0, 0, 0, 0, −3, 0, 0, 0, −3, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 0, −3, 0, 0, 3, 0, 0, 0, 0,
0, 3, 0, 0, 0, 3, −3, 0, 0, 0, 0, −3, 0, 3, 0, 3, 3, 3, $\frac{3}{2}$, $\frac{3}{2}$, 3, 0, 0, −3, −3, 0, 0, 0,
−3, −3, $-\frac{3}{2}$, 0, 0, 0, $\frac{3}{2}$, 0, −3, 0, 0, 0, 0, 0, 3, 3, −3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0,
−3, $-\frac{3}{2}$, −3, 0, $-\frac{3}{2}$, 0, 0, −3, 0, 0, 0, 0, −3, 3, 3, 3, −3, −3, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2,
2, −2, −2, 2, −2, −2, 2, 2, 2, −2, 2, −2, 2, −2, −2, 2, −2, 2, 2, 2, 2, −2, 2, 3, 3,

3, $\frac{3}{2}$, 3, 3, $\frac{3}{2}$, $\frac{3}{2}$, 3, 3, 6, 6, 6, 3, 6, 3, 6, 3, 6, 6, 3, 3, 6, 3, $\frac{3}{2}$, 3, 3, 3, 3, 6, 6,
3, 3, $\frac{3}{2}$, 6, 6, 3, 3, 3, 6, 3, 3, 3, $\frac{3}{2}$, 3, 3, 6, 3, 6, 3, 3, 6, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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−4, 0, 0, 0, 0, −2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, −3, 0, 0, 0, 0, 0, 0, 3, 3, 3, 0, 0,
0, 0, −3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 0, −3, 0, 0, 3, 0, 0, 0, 0, 0,
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0, 0, 0, 0, −3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, −3, 0, 0, 3, −3, −3, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3, 6, 6, 6, 6, 6, 6, 3, 6, 6, 3, 3, 6, 6, 6, 3, 6, 6,
6, 6, 3, 6, 3, 6, 6, 3, 6, 6, 6, 3, 6, 3, 6, 6, 3, 6, 6, 3, 6, 3, 6, 6, 6, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6}

MatrixRank[Transpose[mat]]

27

a + 2 b /. Thread[{a, b, c} → IdentityMatrix[3]]

{1, 2, 0}

r /. W[_a, ___] ⧴ 0

$\frac{1}{2}$ S[1] W[𝕖[1, 1, 1], 𝕖[2, 1, 1]] + S[2] W[𝕖[1, 2, 1], 𝕖[2, 1, 2]]

TStar[Expand[2 r /. W[_a, ___] ⧴ 0], BasisAArrow[4]]

A very large output was generated. Here is a sample of it:

{≪8457≫ + 384 RW[𝕖[1, 1, 8], 𝕖[1, 2, 7], 𝕖[1, 3, 6],
𝕖[1, 4, 5], 𝕖[1, 5, 4], 𝕖[1, 6, 3], 𝕖[1, 7, 2], 𝕖[1, 8, 1]] S[8],
≪137≫, RW[𝕖[1, 1, 1], 𝕖[1, 1, 1], 𝕖[1, 1, 1], 𝕖[1, 1, 1], 𝕖[1, 1, 1],
𝕖[1, 1, 1], 𝕖[1, 1, 1], 𝕖[1, 1, 1]] S[1] + ≪7193≫ + ≪1≫}

Show Less | Show More | Show Full Output | Set Size Limit...