

# The Wirtinger-Alexander Law

September-03-10  
5:59 PM

From <http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/pAHandout.html>

$$\begin{array}{l}
 \begin{array}{c} c \swarrow \\ \nearrow a \\ y \quad x \end{array} \rightarrow \begin{array}{c|cc} a & b & c \\ \hline c & -1 & 1-x & y \end{array} \xrightarrow{\substack{at \\ x=y}} \begin{array}{c|cc} \alpha(h) & \alpha(t) & \alpha(h+1) \\ \hline a & b & c \\ \hline c & -1 & 1-x & x \end{array} \\
 \\
 \begin{array}{c} \swarrow c \\ \nearrow a \\ x \quad y \end{array} \rightarrow \begin{array}{c|cc} a & b & c \\ \hline c & -y & x-1 & 1 \end{array} \text{ so } \begin{array}{c} \swarrow a \\ \nearrow b \\ y \quad x \\ \searrow c \end{array} \mapsto \begin{array}{c|cc} a & b & c \\ \hline c & -y & x-1 & 1 \end{array} \\
 \\
 \begin{array}{c} \swarrow a \\ \nearrow b \\ y \quad x \\ \searrow c \end{array} \mapsto \begin{array}{c|cc} a & b & c \\ \hline c & -1 & \frac{x}{y} & \frac{1}{y} \end{array} \\
 \propto \begin{array}{c|cc} a & b & c \\ \hline c & -1 & \frac{x}{y} & \frac{1}{y} \end{array}
 \end{array}$$

The mystery determinant:  
 $T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|,$   
 $s_i = \text{sign}(\#i), d_i = \text{dir}(\#i),$   
 $S = \text{diag}(s_i d_i), B = T(X^{-S} - I)$   
 $A = \det(I - B).$

$$\xrightarrow{\substack{at \\ x=y}} \begin{array}{c|cc} a & 1 & c \\ \hline c & -1 & 1-x & x \end{array}$$

Question For the standard determinant formulas for the Alexander polynomial, what's  $\tilde{E}(A(e^x)^{-1})$ ?

$$\begin{aligned}
 \tilde{E} \det \underbrace{(M_0 + M_+ e^x + M_- e^{-x})}_M^{-1} &= -x \frac{d}{dx} \log \det(M) \\
 &= -x \text{tr} \left( M^{-1} \frac{d}{dx} M \right) \\
 &= -x \text{tr} M^{-1} (M_+ e^x - M_- e^{-x})
 \end{aligned}$$

$(x^s)' = \frac{s}{x} x^s$   
 $\frac{d}{dx} \log \det(M) = \text{tr} \left( M^{-1} \frac{d}{dx} M \right)$   
 $\text{tr} \left( (M) \cdot \left( \begin{smallmatrix} - \\ \vdots \\ \vdots \\ \vdots \end{smallmatrix} \right)^i \right) = M_{ji}$

So that thing which is inverted is the same thing whose determinant we took.