A contact structure on $M^m$ (must be $2n+1$) is a maximally non-integrable field of hyperplanes ($H_x$) $\iff$ $\alpha \in \Omega^1(M)$ s.t. $d\alpha|_{\alpha^\perp}$ is non-degenerate. $\iff$ $\alpha \wedge \underbrace{d\alpha \wedge \dots \wedge d\alpha}_{n} \neq 0$. $\iff$ maximally integrable ("Legendrian") submanifold has dimension $n$.

<u>"Darboux" Theorem.</u> Locally in coordinates $x_i, y_i, z$,

$$\alpha = \sum_{i=1}^{n} y_i\, dx_i - dz$$

$\Longrightarrow$ "Contact topology" makes sense; impose that all transition maps in atlases be "contactomorphisms".

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$n=1$, $m=3$, $\alpha_{std} = y\,dx - dz$ $\left(\begin{array}{l}\text{nice integrable curves}\\ \text{are } (x, f'(x), f(x))\end{array}\right)$

There's also

$$\alpha_{rot} = dz - y\,dx + x\,dy$$

<u>Exercise</u> These two are contactomorphic

<u>Sol'n</u> $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z + xy \end{pmatrix}$

$T^*\alpha_{rot} = dz +$

---

<u>Thurston–Bennequin number</u>: self-linking relative to the contact-induced framing

– invariant under legendrian isotopy but not under general isotopy.

<u>Maslov number</u>:

$$\begin{array}{ccc} \{H_x\} & \hookrightarrow & TM \\ & \searrow & \downarrow \\ & & M \end{array} \qquad \text{Fix a triviallization:} \qquad \begin{array}{c} \mathbb{R}^5 \\ \downarrow \\ \mathbb{R}^3 \end{array}$$

The Maslov number is the rotation number of a curve relative to some triviallization of the contact plane bundle.

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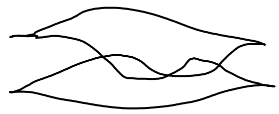
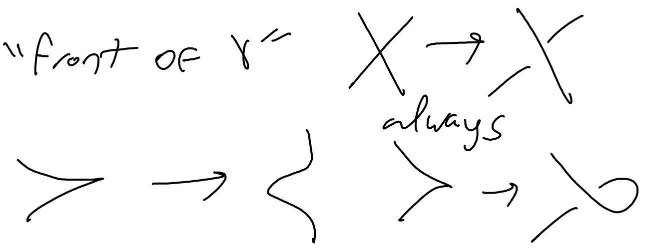
Project a Legendrian curve to the $(x,z)$-plane:

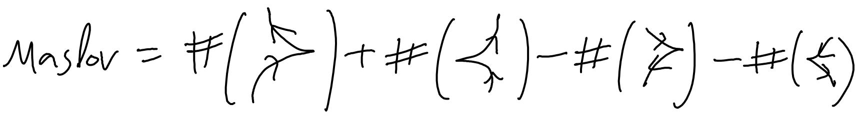


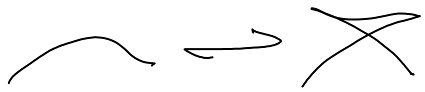
"front of $\gamma$"

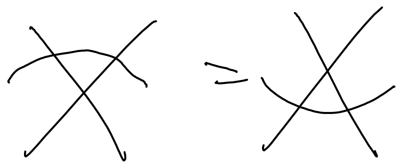



The TB number is the number of crossings (signed) plus the number of right cusps.

The Maslov number is the number of <u>up</u> left or right cusps minus the number of <u>down</u> l/r cusps.

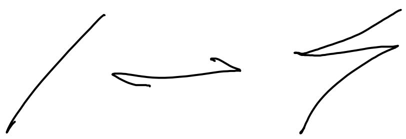



Leg. Reid. moves: 1. 
(complete set of moves!)

2.  3. 

Fuchs & Tabachnikov (97'):

Ordinary knots are the quotient of the above by



Chekanov in 2002 found two Legendrian presentations of $5_2$ that are Legendrian different yet have the same TB & M. [using "Contact Homology"]

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There are also "transverse knots", always transverse to the contact structure.

$\cong$ (conjugacy classes of braids) / just positive Markov moves.

Lenny Ng has a contact knot atlas!