Chicago ax+b Talk as of September 8

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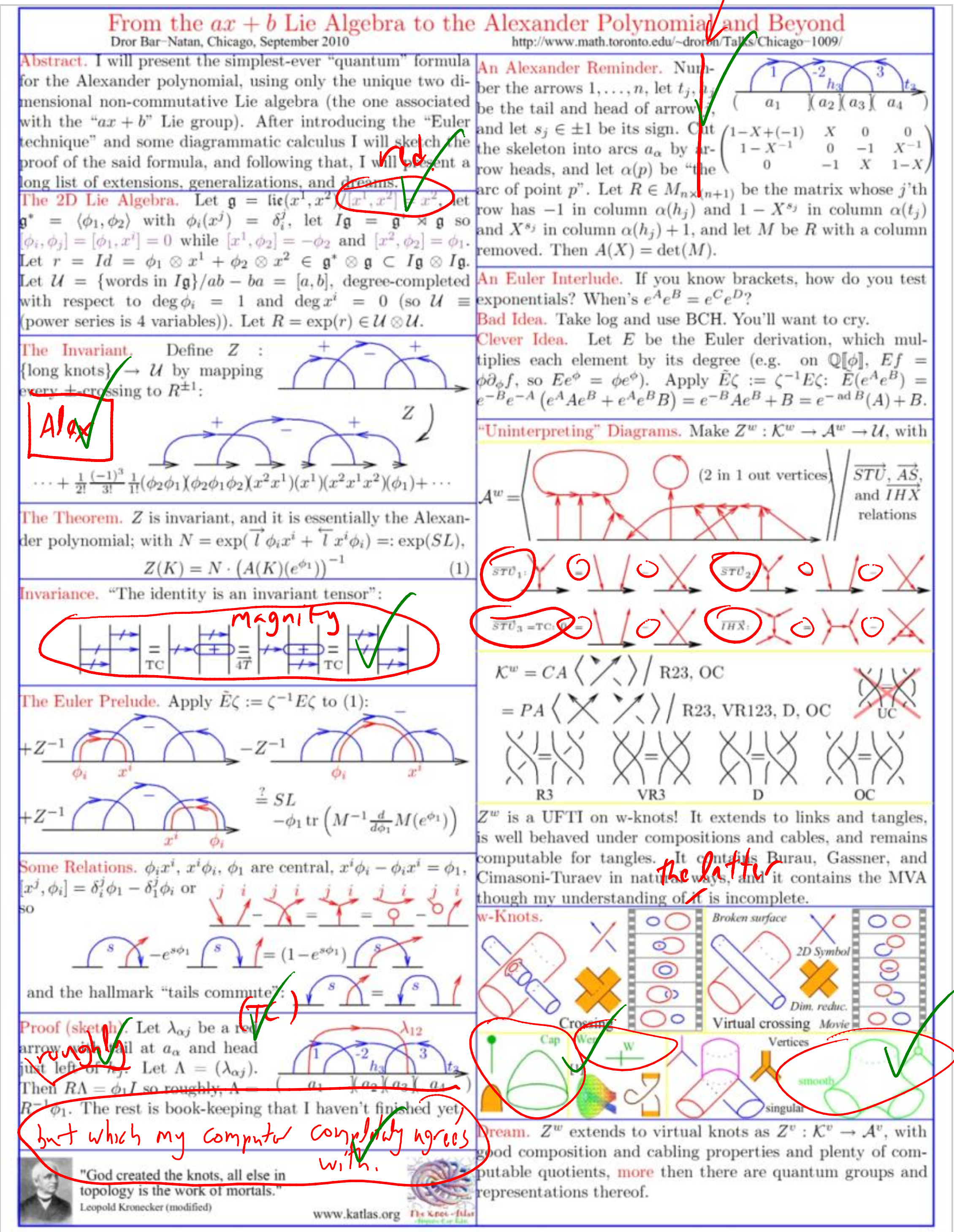

# From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

Dror Bar-Natan, Chicago, September 2010 http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

**Abstract.** I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing the "Euler technique" and some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

**The 2D Lie Algebra.** Let $\mathfrak{g} = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak{g}^* = \langle\phi_1,\phi_2\rangle$ with $\phi_i(x^j) = \delta^j_i$, let $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ so $[\phi_i,\phi_j] = [\phi_1, x^i] = 0$ while $[x^1,\phi_2] = -\phi_2$ and $[x^2,\phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^*\otimes\mathfrak{g} \subset I\mathfrak{g}\otimes I\mathfrak{g}$. Let $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a,b]$, degree-completed with respect to $\deg\phi_i = 1$ and $\deg x^i = 0$ (so $\mathcal{U} \equiv$ (power series is 4 variables)). Let $R = \exp(r) \in \mathcal{U}\otimes\mathcal{U}$.

red

**The Invariant.** Define $Z$ : {long knots} $\to \mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm 1}$:

Alex

$$\cdots + \frac{1}{2!}\frac{(-1)^3}{3!}\frac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1) + \cdots$$

**The Theorem.** $Z$ is invariant, and it is essentially the Alexander polynomial; with $N = \exp(\overleftarrow{l}\,\phi_i x^i + \overleftarrow{l}\,x^i\phi_i) =: \exp(SL)$,

$$Z(K) = N\cdot\left(A(K)(e^{\phi_1})\right)^{-1} \qquad (1)$$

**Invariance.** "The identity is an invariant tensor":

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TC 4T TC

**The Euler Prelude.** Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$ to (1):

$+Z^{-1}$ $\phi_i$ $x^i$ $-Z^{-1}$ $\phi_i$ $x^i$

$+Z^{-1}$ $x^i$ $\phi_i$ $\stackrel{?}{=} SL$ $-\phi_1\operatorname{tr}\left(M^{-1}\frac{d}{d\phi_1}M(e^{\phi_1})\right)$

**Some Relations.** $\phi_i x^i$, $x^i\phi_i$, $\phi_1$ are central, $x^i\phi_i - \phi_i x^i = \phi_1$, $[x^j,\phi_i] = \delta^j_i\phi_1 - \delta^j_1\phi_i$ or so

$-e^{s\phi_1}$ $= (1-e^{s\phi_1})$

and the hallmark "tails commute":

TC

**Proof (sketch).** Let $\lambda_{\alpha j}$ be a red arrow with tail at $a_\alpha$ and head just left of $h_j$. Let $\Lambda = (\lambda_{\alpha j})$. Then $R\Lambda = \phi_1 I$ so roughly, $A = R^{-1}\phi_1$. The rest is book-keeping that I haven't finished yet

rough

but which my computer completely agrees with.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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**An Alexander Reminder.** Number the arrows $1,\ldots,n$, let $t_j, h_j$ be the tail and head of arrow $j$, and let $s_j \in \pm 1$ be its sign. Cut the skeleton into arcs $a_\alpha$ by arrow heads, and let $\alpha(p)$ be "the arc of point $p$". Let $R \in M_{n\times(n+1)}$ be the matrix whose $j$'th row has $-1$ in column $\alpha(h_j)$ and $1 - X^{s_j}$ in column $\alpha(t_j)$ and $X^{s_j}$ in column $\alpha(h_j)+1$, and let $M$ be $R$ with a column removed. Then $A(X) = \det(M)$.

$$\begin{pmatrix} 1-X+(-1) & X & 0 & 0 \\ 1-X^{-1} & 0 & -1 & X^{-1} \\ 0 & -1 & X & 1-X \end{pmatrix}$$

**An Euler Interlude.** If you know brackets, how do you test exponentials? When's $e^Ae^B = e^Ce^D$?

**Bad Idea.** Take log and use BCH. You'll want to cry.

**Clever Idea.** Let $E$ be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\![\phi]\!]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$). Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B) = e^{-B}e^{-A}(e^AAe^B + e^Ae^BB) = e^{-B}Ae^B + B = e^{-\operatorname{ad}B}(A) + B$.

**"Uninterpreting" Diagrams.** Make $Z^w : \mathcal{K}^w \to \mathcal{A}^w \to \mathcal{U}$, with

$\mathcal{A}^w = \langle$ (2 in 1 out vertices) $\rangle / \overrightarrow{STU}, \overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations

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$\overrightarrow{STU}_1$: $\overrightarrow{STU}_2$: $\overrightarrow{STU}_3 = TC$: $\overrightarrow{IHX}$:

$\mathcal{K}^w = CA\langle\ \rangle /$ R23, OC

$= PA\langle\ \rangle /$ R23, VR123, D, OC UC

R3 VR3 D OC

$Z^w$ is a UFTI on w-knots! It extends to links and tangles, is well behaved under compositions and cables, and remains computable for tangles. It contains Burau, Gassner, and Cimasoni-Turaev in natural ways, and the latter it contains the MVA though my understanding of it is incomplete.

**w-Knots.** Crossing Broken surface 2D Symbol Dim. reduc. Virtual crossing Movie Cap Wen W Vertices smooth singular

black.

**Dream.** $Z^w$ extends to virtual knots as $Z^v : \mathcal{K}^v \to \mathcal{A}^v$, with good composition and cabling properties and plenty of computable quotients, more then there are quantum groups and representations thereof.

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Turn all borders green.