Chicago ax+b Talk as of September 7

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From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

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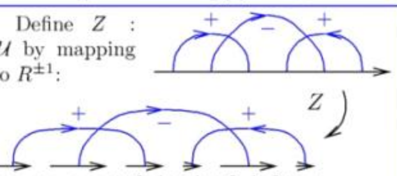
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

The 2D Lie Algebra. Let $\mathfrak{g} = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$ with $\phi_i(x^j) = \delta_i^j$, let $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ so $[\phi_i, \phi_j] = [\phi_1, x^i] = 0$ while $[x^1, \phi_2] = -\phi_2$ and $[x^2, \phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$. Let $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a,b]$, degree-completed with respect to $\deg \phi_i = 1$ and $\deg x^i = 0$ (so $\mathcal{U} \equiv$ (power series is 4 variables)). Let $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$.

The Invariant. Define Z : {long knots} $\to \mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm 1}$:



$$\cdots + \frac{1}{2}\frac{(-1)^3}{3!}\frac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1) + \cdots$$

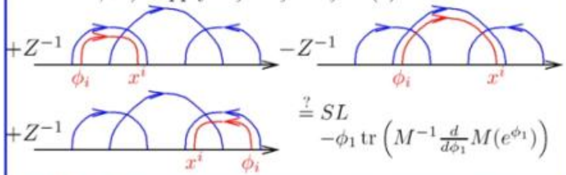
dele

The Theorem. Z is invariant, and it is essentially the Alexander polynomial, with $N = \exp(\overrightarrow{l}\,\phi_i x^i + \overleftarrow{l}\,x^i\phi_i) =: \exp(SL)$,

$$\tilde{Z}(K) = N \cdot \left(A(K)(e^{\phi_1})\right)^{-1}. \qquad (1)$$

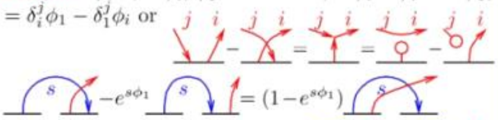
~~Here $A(X) = \det M$ is the Alexander polynomial, where the $n \times n$ (at n crossings) M comes from the Wirtinger presentation and contains only ± 1's and $\pm X^{\pm 1}$'s.~~

~~An~~ Euler Prelude. ~~Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\phi]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$).~~ Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$ to (1):



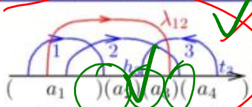
$$\stackrel{?}{=} SL - \phi_1 \operatorname{tr}\left(M^{-1}\tfrac{d}{d\phi_1}M(e^{\phi_1})\right)$$

Some Relations. $\phi_i x^i$, $x^i \phi_i$, ϕ_1 are central, $x^i\phi_i - \phi_i x^i = \phi_1$, $[x^j, \phi_i] = \delta_i^j \phi_1 - \delta_1^j \phi_i$ or so



and the hallmark "tails commute":

Proof (sketch). Cut the skeleton into arcs a_α by arrow heads. Let $\lambda_{\alpha j}$ be a red arrow with tail at a_α and head just left of h_j. Let $\Lambda = (\lambda_{\alpha j})$. Then $M\Lambda = \phi_1 I$ so roughly, $\Lambda = M^{-1}\phi_1$. The rest is book-keeping. that I have not done yet.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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An Alexander interlude.
$A = \det(R'),$ where
$R = \ldots\ldots$

gA

An Euler Interlude
As on blackboard + words
+ BCH

Proof of invariance

"uninterpreting" diagrams
$Z^w : K^w \longrightarrow A^w \longrightarrow U$
⋮
Z is a UFTI of w-knots!
Extends to links and tangles, well behaved under compositions and cables, remains computable for tangles, though I still don't understand links.

What are w-knots?

Vertices, caps, etc

gA

Dream Extends to v-knots and has many polynomial-time & meaningful quotients.

Add "generalizes Burau, Gassner, Cimasoni-Turaev"...