

Pensieve header: Computing the quotient of the envelope of the w-Alexander polynomial by link relations.

```

{s0, s1} = {1, 1};
Gens[_ , -1] = {};
Gens[_ , 0] = {1};
Gens[n_ , m_] := Gens[n, m] = (List @@ Expand[
  (Sum[x[i], {i, n}] + Sum[a[i, j], {i, n}, {j, n}]) ^ m
]) /. (_Integer * b_) => b;
LR1[i_ , j_][Z_] := Module[
  {tails},
  tails = Union[Cases[{Z}, a[k_ , i] => k, Infinity]];
  Expand[Plus[
    If[i == j, s0 x[i] Z, 0],
    Sum[
      (x[i] a[k, j] - x[k] a[i, j]) D[Z, a[k, i]],
      {k, tails}
    ]
  ]
];
LR2[i_ , j_][Z_] := Module[
  {tails, heads},
  tails = Union[Cases[{Z}, a[k_ , j] => k, Infinity]];
  heads = Union[Cases[{Z}, a[j, k_ ] => k, Infinity]];
  Expand[Plus[
    If[i == j, s1 x[i] Z, 0],
    Sum[
      (x[i] a[k, j] - x[k] a[i, j]) D[Z, a[k, j]],
      {k, tails}
    ],
    Sum[
      (x[j] a[i, k] - x[i] a[j, k]) D[Z, a[j, k]],
      {k, heads}
    ]
  ]
];
Rels[n_ , m_] := DeleteCases[
  Union[Flatten[
    Table[
      {LR1[i, j] /@ Gens[n, m - 1], LR2[j, i] /@ Gens[n, m - 1]},
      {i, 2, n}, {j, n}
    ]
  ]],
  0];
RRels[n_ , m_] := Module[
  {gens, rels, mat},
  gens = Gens[n, m];
  rels = Rels[n, m];
  mat = rels /. Thread[gens -> IdentityMatrix[Length[gens]]];
  DeleteCases[RowReduce[mat].gens, 0]
]

```

**Gens[2, 0]**

{1}

**Gens[2, 1]**

{a[1, 1], a[1, 2], a[2, 1], a[2, 2], x[1], x[2]}

**Gens[2, 2]**

{a[1, 1]<sup>2</sup>, a[1, 1] a[1, 2], a[1, 2]<sup>2</sup>, a[1, 1] a[2, 1], a[1, 2] a[2, 1], a[2, 1]<sup>2</sup>, a[1, 1] a[2, 2],  
a[1, 2] a[2, 2], a[2, 1] a[2, 2], a[2, 2]<sup>2</sup>, a[1, 1] x[1], a[1, 2] x[1], a[2, 1] x[1],  
a[2, 2] x[1], x[1]<sup>2</sup>, a[1, 1] x[2], a[1, 2] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]<sup>2</sup>}

**Union[LR1[1, 2] /@ Gens[2, 2]]**

{0, a[1, 1] a[2, 2] x[1] - a[1, 1] a[1, 2] x[2], a[1, 2] a[2, 2] x[1] - a[1, 2]<sup>2</sup> x[2],  
2 a[2, 1] a[2, 2] x[1] - 2 a[1, 2] a[2, 1] x[2], a[2, 2]<sup>2</sup> x[1] - a[1, 2] a[2, 2] x[2],  
a[2, 2] x[1]<sup>2</sup> - a[1, 2] x[1] x[2], a[2, 2] x[1] x[2] - a[1, 2] x[2]<sup>2</sup>}

**LR2[1, 2][a[2, 1]]**

-a[2, 1] x[1] + a[1, 1] x[2]

{LR2[2, 2][a[1, 2]], LR2[2, 2][a[1, 2]]}

{-a[2, 2] x[1] + 2 a[1, 2] x[2], -a[2, 2] x[1] + 2 a[1, 2] x[2]}

**LR2[2, 2] /@ Gens[2, 2]**

{a[1, 1]<sup>2</sup> x[2], -a[1, 1] a[2, 2] x[1] + 2 a[1, 1] a[1, 2] x[2],  
-2 a[1, 2] a[2, 2] x[1] + 3 a[1, 2]<sup>2</sup> x[2], a[1, 1] a[2, 1] x[2],  
-a[2, 1] a[2, 2] x[1] + 2 a[1, 2] a[2, 1] x[2], a[2, 1]<sup>2</sup> x[2],  
a[1, 1] a[2, 2] x[2], -a[2, 2]<sup>2</sup> x[1] + 2 a[1, 2] a[2, 2] x[2], a[2, 1] a[2, 2] x[2],  
a[2, 2]<sup>2</sup> x[2], a[1, 1] x[1] x[2], -a[2, 2] x[1]<sup>2</sup> + 2 a[1, 2] x[1] x[2],  
a[2, 1] x[1] x[2], a[2, 2] x[1] x[2], x[1]<sup>2</sup> x[2], a[1, 1] x[2]<sup>2</sup>,  
-a[2, 2] x[1] x[2] + 2 a[1, 2] x[2]<sup>2</sup>, a[2, 1] x[2]<sup>2</sup>, a[2, 2] x[2]<sup>2</sup>, x[1] x[2]<sup>2</sup>, x[2]<sup>3</sup>}

**Rels[2, 1]**

{x[2]}

**gens = Gens[2, 2]**

{a[1, 1]<sup>2</sup>, a[1, 1] a[1, 2], a[1, 2]<sup>2</sup>, a[1, 1] a[2, 1], a[1, 2] a[2, 1], a[2, 1]<sup>2</sup>, a[1, 1] a[2, 2],  
a[1, 2] a[2, 2], a[2, 1] a[2, 2], a[2, 2]<sup>2</sup>, a[1, 1] x[1], a[1, 2] x[1], a[2, 1] x[1],  
a[2, 2] x[1], x[1]<sup>2</sup>, a[1, 1] x[2], a[1, 2] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]<sup>2</sup>}



**rrels = DeleteCases [RowReduce [mat] . gens, 0]**

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[1, 1] x[1], a[1, 2] x[1], a[2, 1] x[1], a[2, 2] x[1], x[1]^2,$$

$$a[1, 1] x[2], a[1, 2] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

$$\{a[2, 1] x[1], a[2, 2] x[1], a[1, 1] x[2],$$

$$a[1, 2] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

**rrels = RRelS [2, 2]**

$$\{a[2, 1] x[1], a[2, 2] x[1] - 2 a[1, 2] x[2],$$

$$a[1, 1] x[2], a[2, 1] x[2], a[2, 2] x[2], x[1] x[2], x[2]^2\}$$

**Length [gens] - Length [rrels]**

14

**RRelS [2, 3]**

$$\{a[1, 1] a[2, 1] x[1], a[1, 2] a[2, 1] x[1] - a[1, 1] a[1, 2] x[2], a[2, 1]^2 x[1],$$

$$a[1, 1] a[2, 2] x[1] - 2 a[1, 1] a[1, 2] x[2], a[1, 2] a[2, 2] x[1] - \frac{3}{2} a[1, 2]^2 x[2],$$

$$a[2, 1] a[2, 2] x[1], a[2, 2]^2 x[1] - 2 a[1, 2] a[2, 2] x[2], a[2, 1] x[1]^2,$$

$$a[2, 2] x[1]^2 - 2 a[1, 2] x[1] x[2], a[1, 1]^2 x[2], a[1, 1] a[2, 1] x[2],$$

$$a[1, 2] a[2, 1] x[2], a[2, 1]^2 x[2], a[1, 1] a[2, 2] x[2], a[2, 1] a[2, 2] x[2],$$

$$a[2, 2]^2 x[2], a[1, 1] x[1] x[2], a[2, 1] x[1] x[2], a[2, 2] x[1] x[2], x[1]^2 x[2],$$

$$a[1, 1] x[2]^2, a[1, 2] x[2]^2, a[2, 1] x[2]^2, a[2, 2] x[2]^2, x[1] x[2]^2, x[2]^3\}$$

**RRels[2, 4]**

$$\left\{ \begin{aligned} & a[1, 1]^2 a[2, 1] x[1], a[1, 1] a[1, 2] a[2, 1] x[1] - a[1, 1]^2 a[1, 2] x[2], \\ & a[1, 2]^2 a[2, 1] x[1] - a[1, 1] a[1, 2]^2 x[2], a[1, 1] a[2, 1]^2 x[1], \\ & a[1, 2] a[2, 1]^2 x[1], a[2, 1]^3 x[1], a[1, 1]^2 a[2, 2] x[1] - 2 a[1, 1]^2 a[1, 2] a[2, 2] x[2], \\ & a[1, 1] a[1, 2] a[2, 2] x[1] - \frac{3}{2} a[1, 1] a[1, 2]^2 x[2], a[1, 2]^2 a[2, 2] x[1] - \frac{4}{3} a[1, 2]^3 x[2], \\ & a[1, 1] a[2, 1] a[2, 2] x[1], a[1, 2] a[2, 1] a[2, 2] x[1] - a[1, 1] a[1, 2] a[2, 2] x[2], \\ & a[2, 1]^2 a[2, 2] x[1], a[1, 1] a[2, 2]^2 x[1] - 2 a[1, 1] a[1, 2] a[2, 2] x[2], \\ & a[1, 2] a[2, 2]^2 x[1] - \frac{3}{2} a[1, 2]^2 a[2, 2] x[2], a[2, 1] a[2, 2]^2 x[1], \\ & a[2, 2]^3 x[1] - 2 a[1, 2] a[2, 2]^2 x[2], a[1, 1] a[2, 1] x[1]^2, \\ & a[1, 2] a[2, 1] x[1]^2 - a[1, 1] a[1, 2] x[1] x[2], a[2, 1]^2 x[1]^2, \\ & a[1, 1] a[2, 2] x[1]^2 - 2 a[1, 1] a[1, 2] x[1] x[2], a[1, 2] a[2, 2] x[1]^2 - \frac{3}{2} a[1, 2]^2 x[1] x[2], \\ & a[2, 1] a[2, 2] x[1]^2, a[2, 2]^2 x[1]^2 - 3 a[1, 2]^2 x[2]^2, a[2, 1] x[1]^3, \\ & a[2, 2] x[1]^3 - 2 a[1, 2] x[1]^2 x[2], a[1, 1]^3 x[2], a[1, 1]^2 a[2, 1] x[2], \\ & a[1, 1] a[1, 2] a[2, 1] x[2], a[1, 2]^2 a[2, 1] x[2] - \frac{2}{3} a[1, 1] a[1, 2] a[2, 2] x[2], \\ & a[1, 1] a[2, 1]^2 x[2], a[1, 2] a[2, 1]^2 x[2], a[2, 1]^3 x[2], a[1, 1]^2 a[2, 2] x[2], \\ & a[1, 1] a[2, 1] a[2, 2] x[2], a[1, 2] a[2, 1] a[2, 2] x[2], a[2, 1]^2 a[2, 2] x[2], \\ & a[1, 1] a[2, 2]^2 x[2], a[2, 1] a[2, 2]^2 x[2], a[2, 2]^3 x[2], a[1, 1]^2 x[1] x[2], \\ & a[1, 1] a[2, 1] x[1] x[2], a[1, 2] a[2, 1] x[1] x[2], a[2, 1]^2 x[1] x[2], a[1, 1] a[2, 2] x[1] x[2], \\ & a[1, 2] a[2, 2] x[1] x[2] - \frac{3}{2} a[1, 2]^2 x[2]^2, a[2, 1] a[2, 2] x[1] x[2], a[2, 2]^2 x[1] x[2], \\ & a[1, 1] x[1]^2 x[2], a[2, 1] x[1]^2 x[2], a[2, 2] x[1]^2 x[2], x[1]^3 x[2], a[1, 1]^2 x[2]^2, \\ & a[1, 1] a[1, 2] x[2]^2, a[1, 1] a[2, 1] x[2]^2, a[1, 2] a[2, 1] x[2]^2, a[2, 1]^2 x[2]^2, \\ & a[1, 1] a[2, 2] x[2]^2, a[1, 2] a[2, 2] x[2]^2, a[2, 1] a[2, 2] x[2]^2, a[2, 2]^2 x[2]^2, \\ & a[1, 1] x[1] x[2]^2, a[1, 2] x[1] x[2]^2, a[2, 1] x[1] x[2]^2, a[2, 2] x[1] x[2]^2, \\ & x[1]^2 x[2]^2, a[1, 1] x[2]^3, a[1, 2] x[2]^3, a[2, 1] x[2]^3, a[2, 2] x[2]^3, x[1] x[2]^3, x[2]^4 \end{aligned} \right\}$$

**Study[M\_] := Module[**

**{t, r, dims2},**

**dims2 = Table[**

**Print[{2, m} → {t = Length[Gens[2, m]], r = Length[RRels[2, m]], t - r};**

**{t, r, t - r},**

**{m, M}**

**];**

**{dims2, Last /@ dims2, Factor[InterpolatingPolynomial[Last /@ dims2, m]]}**

**];**

**{s0, s1} = {1, 1}; Study[8]**

$\{2, 1\} \rightarrow \{6, 1, 5\}$

$\{2, 2\} \rightarrow \{21, 7, 14\}$

$\{2, 3\} \rightarrow \{56, 26, 30\}$

$\{2, 4\} \rightarrow \{126, 71, 55\}$

$\{2, 5\} \rightarrow \{252, 161, 91\}$

$\{2, 6\} \rightarrow \{462, 322, 140\}$

$\{2, 7\} \rightarrow \{792, 588, 204\}$

$\{2, 8\} \rightarrow \{1287, 1002, 285\}$

$\left\{ \left\{ \{6, 1, 5\}, \{21, 7, 14\}, \{56, 26, 30\}, \{126, 71, 55\}, \right. \right.$   
 $\left. \{252, 161, 91\}, \{462, 322, 140\}, \{792, 588, 204\}, \{1287, 1002, 285\} \right\},$   
 $\left. \{5, 14, 30, 55, 91, 140, 204, 285\}, \frac{1}{6} (1+m) (2+m) (3+2m) \right\}$

**{s0, s1} = {1, -1}; Study[8]**

$\{2, 1\} \rightarrow \{6, 1, 5\}$

$\{2, 2\} \rightarrow \{21, 8, 13\}$

$\{2, 3\} \rightarrow \{56, 30, 26\}$

$\{2, 4\} \rightarrow \{126, 81, 45\}$

$\{2, 5\} \rightarrow \{252, 181, 71\}$

$\{2, 6\} \rightarrow \{462, 357, 105\}$

$\{2, 7\} \rightarrow \{792, 644, 148\}$

$\{2, 8\} \rightarrow \{1287, 1086, 201\}$

$\left\{ \left\{ \{6, 1, 5\}, \{21, 8, 13\}, \{56, 30, 26\}, \{126, 81, 45\}, \{252, 181, 71\}, \{462, 357, 105\}, \right. \right.$   
 $\left. \{792, 644, 148\}, \{1287, 1086, 201\} \right\}, \{5, 13, 26, 45, 71, 105, 148, 201\}, \frac{1}{6} (1+m) (6+8m+m^2) \left. \right\}$

**{s0, s1} = {-1, 1}; Study[8]**

{2, 1} → {6, 1, 5}

{2, 2} → {21, 8, 13}

{2, 3} → {56, 30, 26}

{2, 4} → {126, 81, 45}

{2, 5} → {252, 181, 71}

{2, 6} → {462, 357, 105}

{2, 7} → {792, 644, 148}

{2, 8} → {1287, 1086, 201}

{ { {6, 1, 5}, {21, 8, 13}, {56, 30, 26}, {126, 81, 45}, {252, 181, 71}, {462, 357, 105},  
 {792, 644, 148}, {1287, 1086, 201} }, {5, 13, 26, 45, 71, 105, 148, 201},  $\frac{1}{6} (1 + m) (6 + 8 m + m^2)$  }

**{s0, s1} = {-1, -1}; Study[8]**

{2, 1} → {6, 1, 5}

{2, 2} → {21, 7, 14}

{2, 3} → {56, 24, 32}

{2, 4} → {126, 63, 63}

{2, 5} → {252, 141, 111}

{2, 6} → {462, 282, 180}

{2, 7} → {792, 518, 274}

{2, 8} → {1287, 890, 397}

{ { {6, 1, 5}, {21, 7, 14}, {56, 24, 32}, {126, 63, 63}, {252, 141, 111}, {462, 282, 180},  
 {792, 518, 274}, {1287, 890, 397} }, {5, 14, 32, 63, 111, 180, 274, 397},  $\frac{1}{6} (6 + 17 m + 3 m^2 + 4 m^3)$  }

**Factor**  $\left[ \frac{1}{6} (6 + 17 m + 3 m^2 + 4 m^3) - \frac{1}{6} (6 + 14 m + 9 m^2 + m^3) \right]$

$\frac{1}{2} (-1 + m)^2 m$

**Factor**  $\left[ \frac{1}{6} (6 + 17 m + 3 m^2 + 4 m^3) - \frac{1}{6} (6 + 13 m + 9 m^2 + 2 m^3) \right]$

$\frac{1}{3} (-2 + m) (-1 + m) m$

**Factor**  $\left[ \frac{1}{6} (6 + 13 m + 9 m^2 + 2 m^3) - \frac{1}{6} (6 + 14 m + 9 m^2 + m^3) \right]$

$\frac{1}{6} (-1 + m) m (1 + m)$

```

PrimitivesToFull[p_List] := Module[
  {lp, x, ser},
  lp = Length[p];
  ser = Normal[Series[
    Product[(1 - x^i)^(-p[[i]]), {i, lp}],
    {x, 0, lp}
  ]];
  Table[Coefficient[ser, x, i], {i, 0, lp}]
];
FullToPrimitives[{1}] = {};
FullToPrimitives[{1, mid___, last_}] := Module[{prev},
  prev = FullToPrimitives[{1, mid}];
  Append[
    prev,
    last - Last[PrimitivesToFull[Append[prev, 0]]]
  ]
];
PrimitivesToFull[{4, 0, 0, 0, 0, 0}]
{1, 4, 10, 20, 35, 56, 84}

FullToPrimitives[Table[ $\frac{1}{6} (1+m)(2+m)(3+2m)$ , {m, 0, 20}]]
{5, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

FullToPrimitives[Table[ $\frac{1}{6} (1+m)(6+8m+m^2)$ , {m, 0, 20}]]
{5, -2, 1, -1, 2, -3, 4, -5, 8, -13, 18, -25, 40, -62, 90, -135, 210, -324, 492, -750}

FullToPrimitives[Table[ $\frac{1}{6} (6+17m+3m^2+4m^3)$ , {m, 0, 20}]]
{5, -1, 2, -2, 2, -5, 6, -7, 12, -21, 30, -44, 74, -119, 182, -288, 466, -750, 1198, -1920}

FullToPrimitives[Table[m!, {m, 0, 10}]]
{1, 1, 4, 17, 92, 572, 4156, 34159, 314368, 3199844}

FullToPrimitives[Table[1+m^2, {m, 0, 10}]]
{2, 2, 2, -1, -2, -3, 2, 4, 0, -5}

Study[n_, M_] := Module[
  {t, r, dims},
  dims = Table[
    Print[{n, m} → {t = Length[Gens[n, m]], r = Length[RRels[n, m]], t - r}];
    {t, r, t - r},
    {m, M}
  ];
  {dims, Last /@ dims, Factor[InterpolatingPolynomial[Last /@ dims, m]]}
];
{s0, s1} = {-1, 1}; Study[3, 8]

```

$\{3, 1\} \rightarrow \{12, 2, 10\}$  $\{3, 2\} \rightarrow \{78, 29, 49\}$  $\{3, 3\} \rightarrow \{364, 189, 175\}$  $\{3, 4\} \rightarrow \{1365, 850, 515\}$  $\{3, 5\} \rightarrow \{4368, 3046, 1322\}$