Proj Q

March-03-10
9:45 AM

$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z)$ $\quad\left[=(x\wedge(y\wedge z))\wedge(z\wedge(y\wedge z))=\ldots\ldots\right]$

$x=y,\ z=1 \rightarrow x\wedge x=x\wedge x$

$x=z,\ y=1 \longrightarrow x\wedge x=(x\wedge x)\wedge 1=x\wedge x$

$x=1\ y=z \rightarrow 1=1\wedge(y\wedge y)=1$

... It doesn't seem like $x\wedge x=x$ follows from

$1\wedge x=1$ & $x\wedge 1=x$.

Is it needed? | always, $\bar{a}\wedge 1=\bar{a}$ & $1\wedge\bar{a}=0$

$$0=[(1+\bar{x})\wedge(1+\bar{y})]\wedge(1+\bar{z})-[(1+\bar{x})\wedge(1+\bar{z})]\wedge[(1+\bar{y})\wedge(1+\bar{z})]$$

$$\begin{aligned}
&=1-1 && \text{deg } 0\\
&+\bar{x}-\bar{x} && \text{deg } 1\\
&+\bar{x}\wedge\bar{y}+\bar{x}\wedge\bar{z}-\bar{x}\wedge\bar{z}-\bar{x}\wedge\bar{y} && \text{deg } 2\\
&+(\bar{x}\wedge\bar{y})\wedge\bar{z}-\bar{x}\wedge(\bar{y}\wedge\bar{z})-(\bar{x}\wedge\bar{z})\wedge\bar{y} && \text{deg } 3\\
&-(\bar{x}\wedge\bar{z})\wedge(\bar{y}\wedge\bar{z}) && \text{deg } 4
\end{aligned}$$

No need for $x\wedge x=x$ up to here!

However it is needed in order to show that the bracket is anti-symmetric, $\bar{x}\wedge\bar{x}=0$:

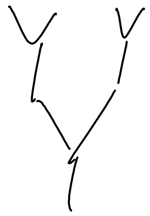
$$0=x\wedge x-x=(1+\bar{x})\wedge(1+\bar{x})-(1+\bar{x})$$

$$\begin{aligned}
&=1-1 && \text{deg } 0\\
&+\bar{x}-\bar{x} && \text{deg } 1\\
&+\bar{x}\wedge\bar{x} && \text{deg } 2
\end{aligned}$$

If we do not impose $x\wedge x=x$, do we get a "Leibnitz algebra"?

Questions If an algebra is Lie at the level of generators, is it Lie in general?

generators, is it Lie in general?

$[[a,b],[a,b]] =$ 

$x\wedge y - \bar{x}\wedge\bar{y} = x\wedge(y-\bar{y}) + (x-\bar{x})\wedge\bar{y} = x\wedge 1 + 1\wedge\bar{y} = x$

$\hookrightarrow = (x-\bar{x})\wedge y + \bar{x}\wedge(y-\bar{y}) = 1\wedge y + \bar{x}\wedge 1 = 1+\bar{x} = x$

Experiment left.

$$((x\wedge y)\wedge z)\wedge w = ((x\wedge y)\wedge w)\wedge(z\wedge w)$$

so

$$((\bar{x}\wedge\bar{y}+x)\wedge(\bar{z}+1))\wedge(\bar{w}+1) = ((\bar{x}\wedge\bar{y}+x)\wedge(\bar{w}+1))\wedge((\bar{z}+1)\wedge(\bar{w}+1))$$

deg 0: $(x\wedge 1)\wedge 1 = (x\wedge 1)\wedge(1\wedge 1)$ ✓

deg 1: $x\wedge\bar{z} + x\wedge\bar{w} = x\wedge\bar{w} + x\wedge\bar{z}$ ✓

deg 2: $\bar{x}\wedge\bar{y} + (x\wedge\bar{z})\wedge\bar{w} = \bar{x}\wedge\bar{y} + (x\wedge\bar{w})\wedge\bar{z} + x\wedge(\bar{z}\wedge\bar{w})$

Aside $((x\wedge y)\wedge z)\wedge w = ((x\wedge y)\wedge(y\wedge z))\wedge w = ((x\wedge y)\wedge w)\wedge((y\wedge z)\wedge w)$

$= \ldots\ldots$

Experiment right.

$(w\wedge x)\wedge(y\wedge z) = (w\wedge(y\wedge z))\wedge(x\wedge(y\wedge z))$ so

$$[(1+\bar{w})\wedge(1+\bar{x})]\wedge(\bar{y}\wedge\bar{z}+\bar{y}+1) = [(1+\bar{w})\wedge(\bar{y}\wedge\bar{z}+\bar{y}+1)]\wedge[(1+\bar{x})\wedge(\bar{y}\wedge\bar{z}+\bar{y}+1)]$$

deg 0: $1 = 1$ ✓

deg 1: $\bar{w} = \bar{w}$ ✓

deg 2: $\bar{w}\wedge\bar{x} + \bar{w}\wedge\bar{y} = \bar{w}\wedge\bar{y} + \bar{w}\wedge\bar{x}$ ✓

deg 3: $\bar{w}\wedge\cancel{(\bar{y}\wedge\bar{z})} + (\bar{w}\wedge\bar{x})\wedge\bar{y} = \bar{w}\wedge\cancel{(\bar{y}\wedge\bar{z})} + (\bar{w}\wedge\bar{y})\wedge\bar{x} + \bar{w}\wedge(\bar{x}\wedge\bar{y})$

deg 4: $(\bar{w}\wedge\bar{x})\wedge(\bar{y}\wedge\bar{z}) = (\bar{w}\wedge(\bar{y}\wedge\bar{z}))\wedge\bar{x} + \bar{w}\wedge(\bar{x}\wedge(\bar{y}\wedge\bar{z})) + (\bar{w}\wedge\bar{y})\wedge(\bar{x}\wedge\bar{y})$

The error from degree 3 fixes degree 4, so Jacobi holds.

Yes 1. Could this have been done in experiment left?

2. Does this generalize?

Experiment left, redone:

$$((w\wedge x)\wedge y)\wedge z = ((w\wedge x)\wedge z)\wedge(y\wedge z) \quad \text{so}$$

$$[(\bar w\wedge\bar x+\bar w+1)\wedge(\bar y+1)]\wedge(\bar z+1) = ((\bar w\wedge\bar x+\bar w+1)\wedge(\bar z+1))\wedge((\bar y+1)\wedge(\bar z+1))$$

deg 0: $1=1$ ✓

deg 1: $\bar w=\bar w$ ✓

deg 2: $\cancel{\bar w\wedge\bar x}+\bar w\wedge\bar y+\bar w\wedge\bar z = \cancel{\bar w\wedge\bar x}+\bar w\wedge\bar z+\bar w\wedge\bar y$ ✓

deg 3: $\cancel{(\bar w\wedge\bar x)}\wedge\bar y+\cancel{(\bar w\wedge\bar x)}\wedge\bar z+(\bar w\wedge\bar y)\wedge\bar z = \cancel{(\bar w\wedge\bar x)\wedge\bar z}+\cancel{(\bar w\wedge\bar x)\wedge\bar y}+(\bar w\wedge\bar z)\wedge\bar y + \bar w\wedge(\bar y\wedge\bar z)$

deg 4: $((\bar w\wedge\bar x)\wedge\bar y)\wedge\bar z = ((\bar w\wedge\bar x)\wedge\bar z)\wedge\bar y+(\bar w\wedge\bar x)\wedge(\bar y\wedge\bar z)+(\bar w\wedge\bar z)\wedge(\bar y\wedge\bar z)$

Works in the same way as experiment right.... As it should, or else proj(free quandle) would not be (free Lie).

Let $[x,y]:=x\wedge y-x$ or $x\wedge y=x+[x,y]$.

We have $[\bar x,\bar y]=\bar x\wedge\bar y - \bar x = \bar x\wedge\bar y-1$

depending on interpretation, with this being the non-bilinear interp.

in the bilinear interp, $[\bar x,\bar y]=\bar x\wedge\bar y$

and also $[\bar x,1]=0$, $[1,\bar x]=0$

$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z) \Rightarrow$

$$[[x,y]+\cancel{x},z]+\cancel{[x,y]}+\cancel{x} = [[x,z]+\underline{x},[y,z]+\underline{y}]+\cancel{[x,z]}+\cancel{x}$$

$\Rightarrow$ $[[x,y],z]=[[x,z],y]+[x,[y,z]]+[[x,z],[y,z]]$

$\Rightarrow$ Same with $\bar x,\bar y,z$. [I may have to introduce a co-product to deal with z].

$$\Delta(\bar{x}\wedge\bar{y}) = \Delta(\bar{x}\wedge y - \bar{x}\wedge 1) = \Delta(x\wedge y - x) = (x\wedge y)\otimes(x\wedge y) - x\otimes x$$

$$(\bar{x}\wedge\bar{y})\otimes 1 + 1\otimes(\bar{x}\wedge\bar{y}) = (x\wedge y - x)\otimes 1 + 1\otimes(x\wedge y - x)$$

$$= (\bar{x}\wedge\bar{y})\otimes(x\wedge y) + x\otimes(\bar{x}\wedge\bar{y}) = (\bar{x}\wedge\bar{y})\otimes 1 + 1\otimes(\bar{x}\wedge\bar{y}) + R \quad \text{with}$$

$$R = (\bar{x}\wedge\bar{y})\otimes(x\wedge y - 1) + (x-1)\otimes(\bar{x}\wedge\bar{y}) = (\bar{x}\wedge\bar{y})\wedge(\bar{x}\wedge\bar{y} + \bar{x}) + \bar{x}\otimes(\bar{x}\wedge\bar{y})$$

It seems that we need the following lemma:

<u>Lemma</u> If $z \in I^m$ (for $m \geq 1$), then $\Delta z - (z\otimes 1 + 1\otimes z) \in \sum_{k+l=m+1} I^k \otimes I^l$

$$\Delta x = x\otimes x \qquad \Delta\bar{x} = x\otimes x - 1\otimes 1 = \bar{x}\otimes x + 1\otimes\bar{x}$$

$$= \bar{x}\otimes 1 + 1\otimes\bar{x} + \bar{x}\otimes\bar{x}$$

$$\Delta(\bar{x}\wedge\bar{y}) = (\bar{x}\otimes 1 + 1\otimes\bar{x} + \bar{x}\otimes\bar{x})\wedge(\bar{y}\otimes 1 + 1\otimes\bar{y} + \bar{y}\otimes\bar{y})$$

$$= (\bar{x}\wedge\bar{y})\otimes 1 + 1\otimes(\bar{x}\wedge\bar{y}) + (\bar{x}\wedge\bar{y})\otimes\bar{x} + \bar{x}\otimes(\bar{x}\wedge\bar{y})$$

$$+ (\bar{x}\wedge\bar{y})\otimes(\bar{x}\wedge\bar{y})$$

<u>Claim</u> With $up: \mathbb{Z}Q\otimes\mathbb{Z}Q \to \mathbb{Z}Q$ the bilinear extension of $x\otimes y \mapsto x\wedge y$, and with $\Delta: \mathbb{Z}Q \to \mathbb{Z}Q\otimes\mathbb{Z}Q$ the linear extension of $x \mapsto x\otimes x$, we have

$$\Delta\circ up = (up\otimes up)\circ(1\otimes\sigma\otimes 1)\circ(\Delta\otimes\Delta) : \mathbb{Z}Q^{\otimes 2} \to \mathbb{Z}Q^{\otimes 2}$$

<u>Proof</u> Both sides of the equation are linear maps, and on (x,y), both give $(x\wedge y)\otimes(x\wedge y)$. □

<u>Proof of lemma</u> By induction on the structure of $z = z_1\wedge z_2$.

Take $z_i \in I^{m_i}$ for $m_i \geq 1$, $i=1,2$. Then

$$\Delta(z) = \Delta(up(z_1\otimes z_2)) = up_2(\Delta(z_1)\otimes\Delta(z_2)) =$$

$$= up_2\left((z_1\otimes 1 + 1\otimes z_1 + z_1'\otimes z_1'')\otimes(z_2\otimes 1 + 1\otimes z_2 + z_2'\otimes z_2'')\right)$$

$$= (z_1\wedge z_2)\otimes 1 + 1\otimes(z_1\wedge z_2)$$

$$+\underbrace{(z_1'\wedge z_2)\otimes z_1''}_{\text{ok}} + \underbrace{z_1'\otimes(z_1''\wedge z_2)}_{\text{ok}} + \underbrace{(z_1'\wedge z_2')\otimes(z_1''\wedge z_2'')}_{\text{doubly ok.}}$$

It remains to show that the bracket is AS in general! $X = x\wedge x \Rightarrow$

$$z = \mu\left(z\otimes 1 + 1\otimes z + \sum z'\otimes z''\right)$$
$$= z + \sum z'\wedge z''$$

Only gives us the uncotrolled $\sum z'\wedge z'' = 0$.

<u>Question</u> If a f.g. algebra satisfies Jacobi and $\bar{x}\wedge\bar{x} = 0$ for every <u>generator</u> $\bar{x}$, does it follow that it is a Lie algebra?

Oops; which Jacobi? w/o AS, there's more than one.

$$[[x,y],[x,y]] = [[[x,y],x],y] + [x,([x,y],y])] = ?$$

Even earlier, can we show that

$$[x,[x,y]] = -[[x,y],x] \quad ? \quad \text{Yes.}$$

Jacobi:

$$[x,[y,z]] \overset{J_x}{=} [[x,y],z] + [y,[x,z]]$$
$$= -[[y,x],z] - [y,[z,x]]$$
$$\overset{J_y}{=} -[[y,x],z] - [[y,z],x] - [z,[y,x]]$$
$$\overset{J_z}{=} -[[y,x],z] - [\cancel{[y,z]}x]] - [\cancel{[z,y]},x] - [y,[z,x]]$$
$$= -[[y,x],z] - [[y,z],x] - [z,[y,x]]$$

This implies ~~$[x,[z,y]] = -[$~~ by setting $x \mapsto x+z$

Nothing

$\Delta\bar{x} = \bar{x}\otimes 1 + 1\otimes\bar{x} + \bar{x}\otimes\bar{x}$. Does this persist? **No:**

Assume true for $z_{1,2}$. Test for $z_1\wedge z_2$:

$\Delta(z_1\wedge z_2) = (\Delta z_1)\wedge(\Delta z_2) =$

$$(z_1\otimes 1+1\otimes z_1+z_1\otimes z_1)\wedge(z_2\otimes 1+1\otimes z_2+z_2\otimes z_2)=$$
$$=(z_1\wedge z_2)\otimes 1+1\otimes(z_1\wedge z_2)+(z_1\wedge z_2)\otimes z_1+z_1\otimes(z_1\wedge z_2)$$
$$+(z_1\wedge z_2)\otimes(z_1\wedge z_2)$$

Yet from the above, with

thus we have AS for z_1 & $z_1\wedge z_2$

$$z_1\wedge z_2=\mathrm{up}(\Delta(z_1\wedge z_2))=z_1\wedge z_2+(z_1\wedge z_2)\wedge z_1+z_1\wedge(z_1\wedge z_2)$$
$$+(z_1\wedge z_2)\wedge(z_1\wedge z_2)$$

[this holds only for $z_{1,2}\in I'$]

(we had that before anyway)

Can we likewise prove that always things obey AS with their conjugates? Would this be enough? Can we prove that z_1 is AS with $z_2\wedge z_3$?

In general,

$\mathrm{up}(\Delta(z_1+z_2))=$ //