Continues
2009-04/Glasgow Handout

Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots

Dror Bar–Natan, Trieste May 2009, http://www.math.toronto.edu/~drorbn/Talks/Trieste–0905

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

Convolutions statement. Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j:\mathfrak{g}\to\mathbb{R}$ be the Jacobian of the exponential map $\exp:\mathfrak{g}\to G$, and let $\Phi:\mathrm{Fun}(G)\to\mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=j^{1/2}(x)f(\exp x)$. Then if $f,g\in\mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f)\star\Phi(g)=\Phi(f\star g).$$

Group-Ring statement. There exists $\omega^2\in\mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi,\psi\in\mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$: (shhh, $\omega^2=j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2(x+y)e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2(x)\omega^2(y)e^xe^y.$$

Properly Complete $\mathcal{U}(\mathfrak{g})$.

Unitary statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ and a (infinite order) unitary ($V^{-1}=V^*$) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\omega(x+y)=\omega(x)\omega(y)$ and so that when $\mathcal{U}(\mathfrak{g})$-valued functions are allowed,

$$V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V.$$

Algebraic statement. With $I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g}$, with $c:\mathcal{U}(I\mathfrak{g})\to\mathcal{U}(I\mathfrak{g})/\mathcal{U}(\mathfrak{g})=\mathcal{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\mathcal{U}(I\mathfrak{g})$, with W the automorphism of $\mathcal{U}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\mathcal{U}(I\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ there exist $\omega\in\mathcal{S}(\mathfrak{g}^*)$ and $V\in\mathcal{U}(I\mathfrak{g})^{\otimes 2}$ so that $V^{-1}=V^*:=SWV$, $c(V\Delta(\omega))=\omega\otimes\omega$, and in $\mathcal{U}(I\mathfrak{g})^{\otimes 2}\otimes\mathcal{U}(\mathfrak{g})$,

$$V(\Delta\otimes 1)(R)=R^{13}R^{23}V.$$

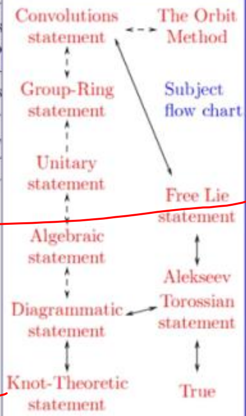
Diagrammatic statement.

For all statements, write conditions w/ parallel numbering:
1. $e^{x+y}V=Ve^xe^y$ - - - - R4
2. $V^*V=I$ - - - -
3. $V\omega(x+y)=\omega(x)\omega(y)$ - - -

Knot-Theoretic statement. There exists a homomorphic expansion Z for w-tangled trivalent graphs.

A full description of w-knots should come here

The Orbit Method. By Fourier analysis, the charcters of $(\mathrm{Fun}(\mathfrak{g})^G,\star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G,\star)$.




Kashiwara


Vergne

add an AMPT statement

Free Lie statement. There exist convergent Lie series F and G so that

$$x+y-\log e^ye^x=(1-e^{-\mathrm{ad}\,x})F+(e^{\mathrm{ad}\,y}-1)G$$

$$\mathrm{tr}(\mathrm{ad}\,x)\partial_xF+\mathrm{tr}(\mathrm{ad}\,y)\partial_yG=\frac{1}{2}\mathrm{tr}\left(\frac{\mathrm{ad}\,x}{e^{\mathrm{ad}\,x}-1}+\frac{\mathrm{ad}\,y}{e^{\mathrm{ad}\,y}-1}-\frac{\mathrm{ad}\,z}{e^{\mathrm{ad}\,z}-1}-1\right)$$

Alekseev-Torossian statement. There is an element $F\in\mathrm{TAut}_2$ with

$$F(x+y)=\log e^xe^y$$

and $j(F)\in\mathrm{im}\,\tilde\delta\subset\mathrm{tr}_2$, where for $a\in\mathrm{tr}_1$, $de\tilde{l}ta(a):=a(x)+a(y)-a(\log e^xe^y)$.


Alekseev


Torossian

The dictionary with ribbon 2-knots should come here.

Unitary $\Longrightarrow$ Group-Ring.

$$\begin{aligned}
&\int \omega^2(x+y)e^{x+y}\phi(x)\psi(y) \\
&= \left\langle \omega(x+y), \omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y)\right\rangle \\
&= \left\langle V\omega(x+y), V\omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y)\right\rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^{x}}\widehat{e^{y}}V\omega(x+y)\phi(x)\psi(y)\rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^{x}}\widehat{e^{y}}\omega(x)\omega(y)\phi(x)\psi(y)\rangle \\
&= \int \omega^2(x)\omega^2(y)e^x e^y\phi(x)\psi(y).
\end{aligned}$$

"abriged" version only, add "tangential". commutes w/ invariants.

Draft

Further boxes;
* convolutions and group ring
* Diff op and Algebraic
* Algebraic and Diagrammatic
* Grrr
* Homomorphic Expansions
* Diagrammatic and A-T

Unitary $\Longrightarrow$ Group-Ring. $\int \omega^2(x+y)e^{x+y}\phi(x)\psi(y)$

$$
\left.\begin{aligned}
&= \left\langle \omega(x+y), \omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle \\
&= \left\langle V\omega(x+y), V\omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}V\omega(x+y)\phi(x)\psi(y) \rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}\omega(x)\omega(y)\phi(x)\psi(y) \rangle
\end{aligned}\right\}
$$

Keep as unabridged but clean

$$
= \int \omega^2(x)\omega^2(y)e^x e^y \phi(x)\psi(y).
$$

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$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

Switch to subscript notation

Group-Ring statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2(x+y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2(x)\omega^2(y)e^x e^y.$$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and a (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega(x+y) = \omega(x)\omega(y)$

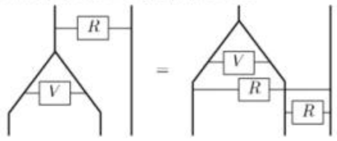
Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $c(V\Delta(\omega)) = \omega \otimes \omega$

Diagrammatic statement. Let $R = \exp \text{⊣} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

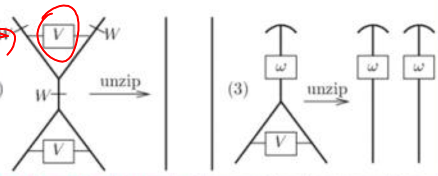
(1)



reorder

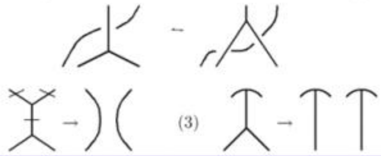
upside down

(2) (3)



Knot-Theoretic statement. There exists a homomorphic expansion Z for w-tangled trivalent graphs. In particular, Z should satisfy $R4$ and intertwine annulus and disk unzips:

(1) (2) (3)



The Orbit Method. By Fourier analysis, the charcters of $(\mathrm{Fun}(\mathfrak{g})^G, \star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G, \star)$.

Kashiwara

Vergne

Measure theoretic statement. Ignoring all ω's, there exists a measure preserving and orbit preserving transformation $T : \mathfrak{g}_x \times \mathfrak{g}_y \to \mathfrak{g}_x \times \mathfrak{g}_y$ for which $e^{x+y} \circ T = e^x e^y$.

Free Lie statement. There exist convergent Lie series F and G so that

$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y} - 1)G$$

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and $j(F) \in \operatorname{im} \tilde{\delta} \subset \mathrm{tr}_2$, where for $a \in \mathrm{tr}$, $\tilde{delta}(a) := a(x) + a(y) - a(\log e^x e^y)$.

Alekseev

Torossian

re-arrange space

Description of w-knots & dictionary with ribbon 2-knots should come here

or A here ?

Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots, Page 2

Unitary $\Longrightarrow$ Group-Ring. $\iint \omega_{x+y}^2 e^{x+y}\phi(x)\psi(y)$

$$
\begin{aligned}
&= \left\langle \omega_{x+y}, \omega_{x+y} e^{x+y}\phi(x)\psi(y) \right\rangle = \left\langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y} \right\rangle \\
&= \langle \omega_x\omega_y, e^x e^y V\phi(x)\psi(y)\omega_{x+y} \rangle = \langle \omega_x\omega_y, e^x e^y \phi(x)\psi(y)\omega_x\omega_y \rangle \\
&= \iint \omega_x^2\omega_y^2 e^x e^y \phi(x)\psi(y).
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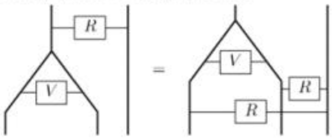
(1)



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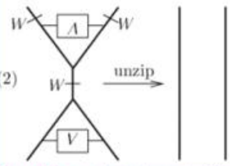
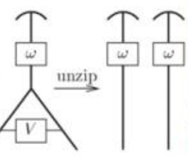


(3)



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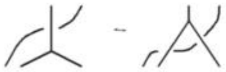
(1)



(2)

(3)

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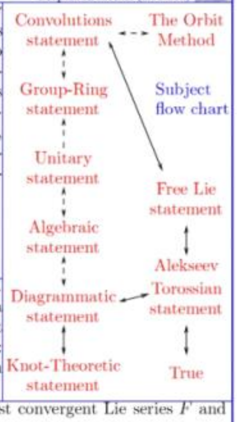
Kashiwara

Vergne



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Alekseev

Alekseev-Torossian statement. There is an element $F\in\mathrm{TAut}_2$ with

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and $j(F)\in\operatorname{im}\tilde\delta\subset\mathrm{tr}_2$, where for $a\in\mathrm{tr}_1$, $\tilde\delta(a):=a(x)+a(y)-a(\log e^xe^y)$.

Torossian

Convolutions and Group Rings (ignoring all Jacobians). If G is finite, $(\mathrm{Fun}(G),\star)\cong(\mathbb{R}G,\cdot)$ via $T:f\mapsto\sum f(a)\tau(a)$. For Lie $\mathfrak{g}$ and G,

$$\begin{array}{ccc} (\mathfrak{g},+)\ni x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{S}}(\mathfrak{g}) \\ \big\downarrow{\scriptstyle\exp} & & \big\downarrow{\scriptstyle\chi} \\ (G,\cdot)\ni e^x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad\text{so}\quad \begin{array}{ccc} \psi\in\mathrm{Fun}(\mathfrak{g}) & \xrightarrow{T} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \big\downarrow{\scriptstyle\Phi^{-1}} & & \big\downarrow{\scriptstyle\chi} \\ \mathrm{Fun}(G) & \xrightarrow{T} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $T\psi=\int\psi(x)e^xdx\in\hat{\mathcal{S}}(\mathfrak{g})$ and $T\Phi^{-1}\psi=\int\psi(x)e^x\in\hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

$\star$ in G: $\iint\psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint\psi_1(x)\psi_2(y)e^{x+y}$

Unitary $\Longrightarrow$ Group-Ring. $\iint\omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

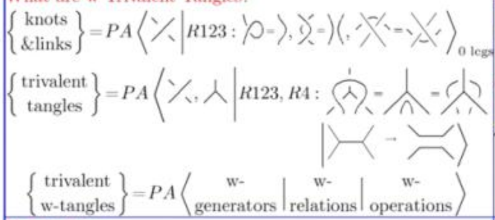
$$=\langle\omega_{x+y},\omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle=\langle V\omega_{x+y},Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$=\langle\omega_x\omega_y,e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle=\langle\omega_x\omega_y,e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$=\iint\omega^2_x\omega^2_ye^xe^y\phi(x)\psi(y).$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

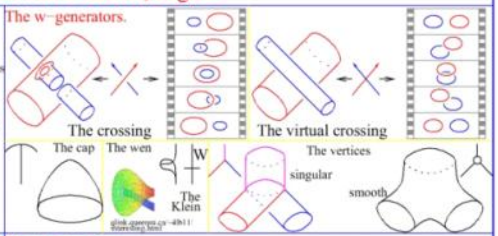
- $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator.
- $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad}x$: $(x\varphi)(y):=\varphi([x,y])$.

Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots, Page 2

What are w-Trivalent Tangles?



The w–generators.



The w–relations.

The w–operations.

Further boxes:

✓ * ~~A~~

✓ * ~~Diagrammatic & Algebraic~~

✓ * ~~Greek~~

✓ * ~~homomorphic expansions~~

* Relation with A-T,

✓ * ~~Relation with Duflo and Fourier analysis~~