Khovanov in Trieste

May-18-09
8:02 AM

Day 1

Homological Algebra $\longrightarrow$ invariants of braids and tangles.

Following "notes on link homology" by Asaeda & Khovanov.

A-mod : left A-module

mod-A - - -

Projective module – a direct summand P of a free module:

$$P \oplus Q \cong A^n$$

$$\begin{tikzcd} & & P \arrow[d] \arrow[dll, dashed, "\exists"'] & \\ M \arrow[rr, two heads] & & N \arrow[r] & 0 \end{tikzcd}$$

If $e \in A$ satisfies $e^2 = e$ then

$A \cong Ae \oplus A(1-e)$ so $A \supset Ae$ is projective in A-mod. [likewise for eA and mod-A]

If $e_{1,2}$ are idempotents,

$$\mathrm{Hom}_A(Ae_1, Ae_2) \cong e_1 A e_2$$

by $f \longmapsto f(e_1)$

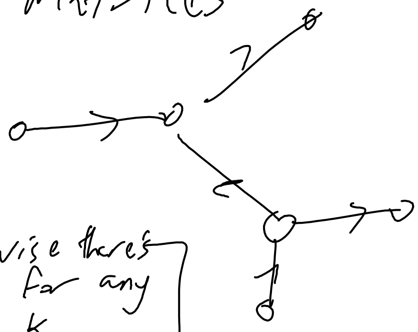
$A \cong eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e)$:

$$\begin{tikzcd} e \arrow[loop left, "eAe"] \arrow[r, bend right, "eA(1-e)"'] & 1-e \arrow[l, bend right, "(1-e)Ae"'] \arrow[loop right, "(1-e)A(1-e)"] \end{tikzcd}$$

In general, if $1 = \sum e_i$ with $e_i e_j = \delta_{ij} I$ then

$A = \bigoplus A e_i$ in A-mod $\qquad A = \bigoplus e_i A$ in mod-A

and $A = \bigoplus e_i A e_j$ "matrices"



Γ - oriented graph

$\rightsquigarrow$ a ring $\mathbb{Z}[\Gamma]$

"path ring" := $\quad$ [likewise there's $k\Gamma$ for any field k]

= Free Abelian group generated by all orientation-respecting paths in Γ, including paths of length 0;

Multiplication by concatenation when possible, 0 otherwise.

Example $\quad a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d$

paths of len $= 0$: $\quad a \quad b \quad c \quad d$

$= 1$: $\quad \alpha \quad \beta \quad \gamma$

$= 2$: $\quad \alpha\beta \;,\; \beta\gamma$

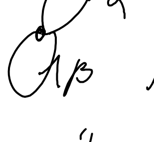
$= 3$: $\quad \alpha\beta\gamma$

≥ 4 : $\quad$ none

The idempotents are a, b, c, d {& they're orthogonal}

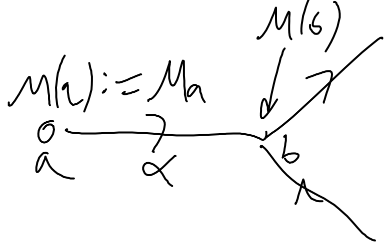
So in general, $\mathbb{Z}(\Gamma)$ is a unital associative ring.

Example

$\mathbb{Z}\Big($  $\Big) = \mathbb{Z}\langle \alpha, \beta \rangle$

$\mathbb{Z}(\Gamma)$ is infinite dimensional iff Γ has directed cycles.

What's mod-$\mathbb{Z}(\Gamma)$? If M's $\in$ mod,



$M = \bigoplus M(a_i)$

Abelian groups

$\alpha : M(a) \longrightarrow M(b)$ by $ma \to m\alpha b = mb$

So a right $\mathbb{Z}(\Gamma)$ module is a set of Abelian groups, one for each vertex, and a set of maps, one for each edge.

Left modules are the same, with the arrows reversed.

$\mathbb{Z}[\Gamma = \underset{a}{\circ} \xrightarrow{\alpha} \underset{b}{\circ} \xrightarrow{\beta} \underset{c}{\circ} \xrightarrow{\gamma} \underset{d}{\circ}]$ is the ring of upper-triangular 4×4 matrices:

$a = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \ldots \qquad \alpha = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \ldots$

Exercise Always, $\mathbb{Z}(\Gamma) \subset Mat(n, \mathbb{Z})$ w/ $n = |V(\Gamma)|$ iff Γ has no oriented cycles.

Remark $\mathbb{K}(\Gamma)$ is a "hereditary algebra", meaning that a submodule of a projective module is projective.

Introduce relations; e.g. "any path of len ≥ 2 is 0"

$\mathbb{Z}(\Gamma) / (\circ \to \circ \to \circ = 0)$ is complexes!

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$$\cdots \circ \to \circ \to \circ \to \circ \to \circ \cdots$$

Bicomplexes: $n \geq 3$

$$A_n \qquad \begin{tikzcd} \underset{1}{\circ} \arrow[r, shift left] & \underset{2}{\circ} \arrow[l, shift left] \arrow[r, shift left] & \underset{3}{\circ} \arrow[l, shift left] & \cdots \arrow[r, shift left] & \underset{n}{\circ} \arrow[l, shift left] \end{tikzcd}$$

around vertex # i:

$$\begin{tikzcd} \underset{i}{\circ} \arrow[r, bend right, "(i|i+1)"'] & \underset{i+1}{\circ} \arrow[l, bend right, "(i+1|i)"'] \arrow[r, bend right] & \underset{i+2}{\circ} \arrow[l, bend right] \end{tikzcd}$$

mod the relations: $\left.\begin{array}{l} (i|i+1|i+2) = 0 \\ (i+2|i+1|i) = 0 \end{array}\right\}$ going twice in any direction is zero.

$X_i = (i|i+1|i) = (i|i-1|i)$ for $1 < i < n$

$$A_n = \left\{ \begin{array}{ll} n \text{ vertices} & \\ 2(n-1) & \text{arrows of length } 1 : (i|i+1), (i+1|i) \\ X_i & \text{for } 1 \leq i \leq n \end{array} \right\}$$

Any path of length ≥ 3 is 0.

$A_2 : \mathbb{Z}\left(\circ \rightleftarrows \circ \right) \Big/ \begin{array}{l} \text{pathes of} \\ \text{len} \geq 3 \end{array}$

$A_1 := \mathbb{Z}[X_1] / X_1^2 = 0$

<u>Claim</u> B_{n+1} acts on complexes of modules over A_n (to be shown later)

Modules on A_n are bi-complexes:

$$\circ \underset{\partial_1}{\overset{\partial_2}{\rightleftarrows}} \circ \underset{\partial_1}{\overset{\partial_2}{\rightleftarrows}} \circ \quad \cdots$$

w/ $\partial_1^2 = \partial_2^2 = 0 \qquad \partial_1\partial_2 = \partial_2\partial_1$

Day 2 . . . .

3 & 4 skipped.

Conference talk

The Hecke algebra:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$
$$T_i T_j = T_j T_i \quad \text{for } |i-j|>1$$
$$T_i^2 = (q-1)T_i + q$$

Realization: $F = \mathbb{F}_q$ = field with $q = p^n$ elements

$$Fl_n = \{ L = (0 \subset L_1 \subset \dots \subset L_{n-1} \subset F^n) : \dim L_i = i \}$$

"flags"

$$T_i L := \sum_{L'_i \neq L_i} L' \qquad L' = (0 \subset L_1 \subset \dots \subset L'_i \subset L_{i+1} \subset \dots)$$

$t = \sqrt{q} \qquad b_i := t^{-1}(1 + T_i) \implies$

$$b_i^2 = (t + t^{-1}) b_i$$
$$b_i b_j = b_j b_i \qquad |i-j|>1$$
$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i$$