Convolutions and Group-Ring

April-05-09
10:37 AM

$$\phi * \psi = \Phi\left(\Phi^{-1}(\phi) * \Phi^{-1}(\psi)\right) \qquad \phi, \psi \in \mathrm{Fun}(\mathfrak{g})$$

$$\Phi^{-1}(\phi * \psi) = \Phi^{-1}(\phi) * \Phi^{-1}(\psi) \quad \text{in } \mathrm{Fun}(G)$$

would-be proof scheme:

$$\begin{tikzcd} \mathrm{Fun}(\mathfrak{g}) \arrow[d] & \mathrm{Fun}(G) \arrow[d] \\ \mathcal{M}(\mathfrak{g}) \arrow[d] & \mathcal{M}(G) \arrow[d] \\ S(\mathfrak{g}) & U(\mathfrak{g}) \end{tikzcd}$$

all verticals are algebra maps.

$$\begin{tikzcd} & \phi,\psi \quad \mathrm{Fun}(\mathfrak{g}) \arrow[r, bend left, "\Phi^{-1}"] \arrow[d, "\cdot dx"] & \mathrm{Fun}(G) \arrow[l, "j^{1/2}\exp^* = \Phi"'] \arrow[d, "\cdot d\mu"] & \\ \mathfrak{g} \arrow[d] & \mathcal{M}(\mathfrak{g}) \arrow[r, "w\exp_*"] \arrow[d] & \mathcal{M}(G) \arrow[d] & G \arrow[d] \\ S(\mathfrak{g}) & \mathcal{M}(S(\mathfrak{g})) \arrow[r, "w\chi_*"] \arrow[d, "b"'] & \mathcal{M}(U(\mathfrak{g})) \arrow[d, "b"] & U(\mathfrak{g}) \\ & S(\mathfrak{g}) \arrow[r, "w\chi"] & U(\mathfrak{g}) & \end{tikzcd}$$

$\mathrm{Fun}(G)$ row: eqn$_1$ [The convolution eqn]

$U(\mathfrak{g})$ row: eqn$_2$ [The group-ring eqn]

diagram is commutative, verticals are algebra maps, horizontals need not be so.

Convolutions and Group Rings (ignoring all Jacobians)

CFL If G is finite, $(\mathrm{Fun}(G), *) \cong (\mathbb{R}G, \cdot)$ by $f \overset{T}{\longmapsto} \sum f(a)\cdot a$

gp	grg	embedding
$(\mathfrak{g}, +)$	$\hat{S}\mathfrak{g}$	$x \overset{\upsilon}{\longmapsto} e^x$
$\downarrow \exp$	$\downarrow \chi$	$\downarrow \exp \qquad \downarrow \chi$
$(G, \cdot)$	$\hat{U}(\mathfrak{g})$	$e^x \xrightarrow{T_\upsilon} e^x$

Convolutions and Group Rings (ignoring all Jacobians)

CFL If G is Finite, $(Fun(G), *) \cong (\mathbb{R}G, \cdot)$ by $f \xmapsto[T]{} \sum f(a)\cdot a$

$$\begin{array}{c|c|c} \begin{array}{c} \text{gp} \\ (\mathfrak{g}, +) \\ \downarrow \exp \\ (G, \cdot) \end{array} & \begin{array}{c} \text{gp rg} \\ \hat{S}\mathfrak{g} \\ \downarrow \chi \\ \hat{\mathcal{U}}(\mathfrak{g}) \end{array} & \begin{array}{c} \text{embedding} \\ x \xmapsto[T]{} e^x \\ \downarrow \exp \quad \downarrow \chi \\ e^x \xrightarrow{T} e^x \end{array} \end{array}$$

$\psi_1, \psi_2 \in Fun(\mathfrak{g}) \rightsquigarrow$ Compare $\Phi^{-1}(\psi_1) * \Phi^{-1}(\psi_2)$ with $\Phi^{-1}(\psi_1 * \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

$(T\Phi^{-1}\psi_i) = \int \psi_i(x) e^x dx \in \hat{\mathcal{U}}(\mathfrak{g})$

Convolution in G:

$$\iint \psi_1(x)\psi_2(y) e^x e^y dx\, dy$$

convolution in $\mathfrak{g}$:

$$\iint \psi_1(x)\psi_2(y) e^{x+y} dx\, dy$$

$$\begin{array}{ccc} \psi_i \in Fun(\mathfrak{g}) & \xrightarrow{T} & \hat{S}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ Fun(G) & \xrightarrow{T} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$