$\omega\epsilon\beta$:= http://www.math.toronto.edu/drorbn/Talks/Montreal-1306

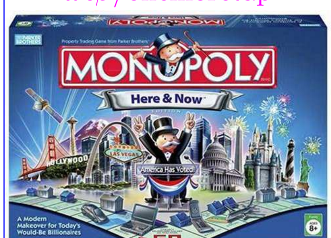
A Quick Introduction to Khovanov Homology

Dror Bar–Natan,
Montreal, June 2013

Why Bother?

$\omega\epsilon\beta$/webcrafterz

$\omega\epsilon\beta$/onemoretap



$\omega\epsilon\beta$/cheat-pcgame

Story/Theorem. To every finite-dimensional metrized Lie algebra $\mathfrak{g}$ and a list $R_1,\dots,R_n$ of representations thereof, there is an associated invariant of n-component links, valued in Laurent polynomials in a variable q (really, in finite linear combinations of characters on the space of metrics on $\mathfrak{g}$).

Queries. Do you know how to tell/prove it? Really? With all the details? Could you teach the story/proof leaving no black boxes in a one-semester course on knot theory to students who are not already experts on Lie algebras? Do I really need to know about Cartan subalgebras and root and weight spaces? Is this theorem at all true?

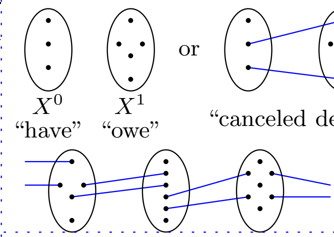
What is Categorification=Concretization=de-abstraction? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things...

...categorification is choosing which 3 it is!

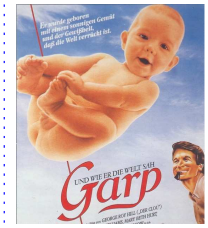
$\mathbb{N}$. Natural numbers $\mapsto$ finite sets, equalities $\mapsto$ bijections, inequalities $\mapsto$ injections and surjections:

$$\binom{2n}{n}=\sum\binom{n}{k}^2\mapsto$$
$$\binom{X\times\{1,2\}}{|X|}\leftrightarrow\bigcup\binom{X}{k}\times\binom{X}{k}.$$

$\mathbb{Z}$. Negative numbers:



The Philosophy Corner



Weaker Categorification. Do the same in the category of vector spaces: "3" becomes V s.t. $\dim V=3$, or better, $V^\bullet=(\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$ and $\chi(V^\bullet):=\sum(-1)^r\dim V^r=3=\sum(-1)^r\dim H^r$. Equalities become homotopies between complexes.

Categorifying $\mathbb{Z}[q^{\pm1}]$. $f=\sum a_jq^j$ becomes $V=\bigoplus V_j$ s.t. $q\dim V:=\sum q^j\dim V_j=f$, or better, $V^\bullet=(\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$, $\deg d=0$, and $\chi_q(V^\bullet):=\sum(-1)^r q\dim V^r=f=\sum(-1)^r q\dim H^r$.

Note. Setting $V\{l\}_j:=V_{j-l}$, we get $q\dim V\{l\}=q^l\, q\dim V$.

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;

$$V=\operatorname{span}\langle v_+,v_-\rangle;\quad \deg v_\pm=\pm1;\quad q\dim V=q+q^{-1};$$

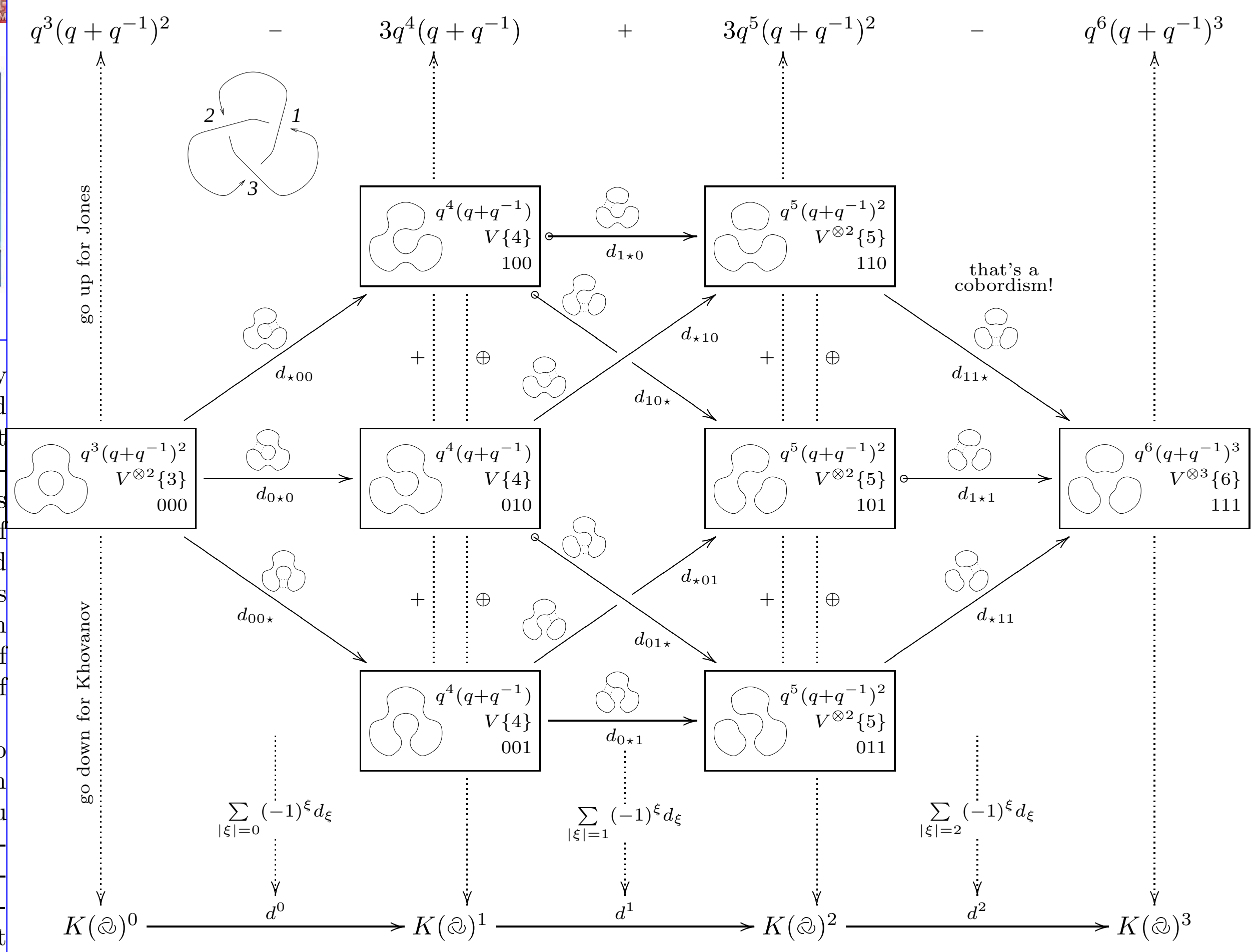
$$K(\bigcirc^k)=V^{\otimes k};\quad K(\times)=\text{Flatten}\Big(0\to \underset{\text{height }0}{K()(\{1\}}\to \underset{\text{height }1}{K(\asymp)\{2\}}\to 0\Big);$$

$$K(\times)=\text{Flatten}\Big(0\to \underset{\text{height }-1}{K(\asymp)\{-2\}}\to \underset{\text{height }0}{K()(\{-1\}}\to 0\Big);$$

$$(\ \cdots\ )\longrightarrow(V\otimes V\xrightarrow{m}V)\qquad m:\begin{cases}v_+\otimes v_-\mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$$

$$(\ \cdots\ )\longrightarrow(V\xrightarrow{\Delta}V\otimes V)\qquad \Delta:\begin{cases}v_+\mapsto v_+\otimes v_-+v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$$

Example: $\qquad = \quad q+q^3+q^5-q^9.$



(here $(-1)^\xi:=(-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$) $\qquad = \quad K(\cdot)$.

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\operatorname{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\operatorname{Kh}(L)$ is strictly stronger than $J(L)$: $J(\bar5_1)=J(10_{132})$ yet $\operatorname{Kh}(\bar5_1)\neq\operatorname{Kh}(10_{132})$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my http://www.math.toronto.edu/~drorbn/papers/Categorification/.

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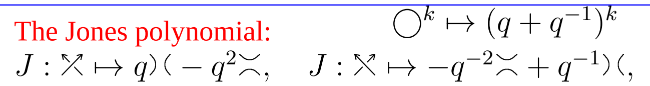
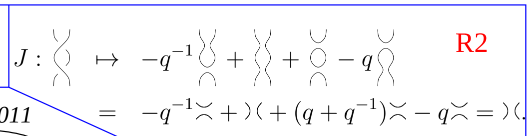
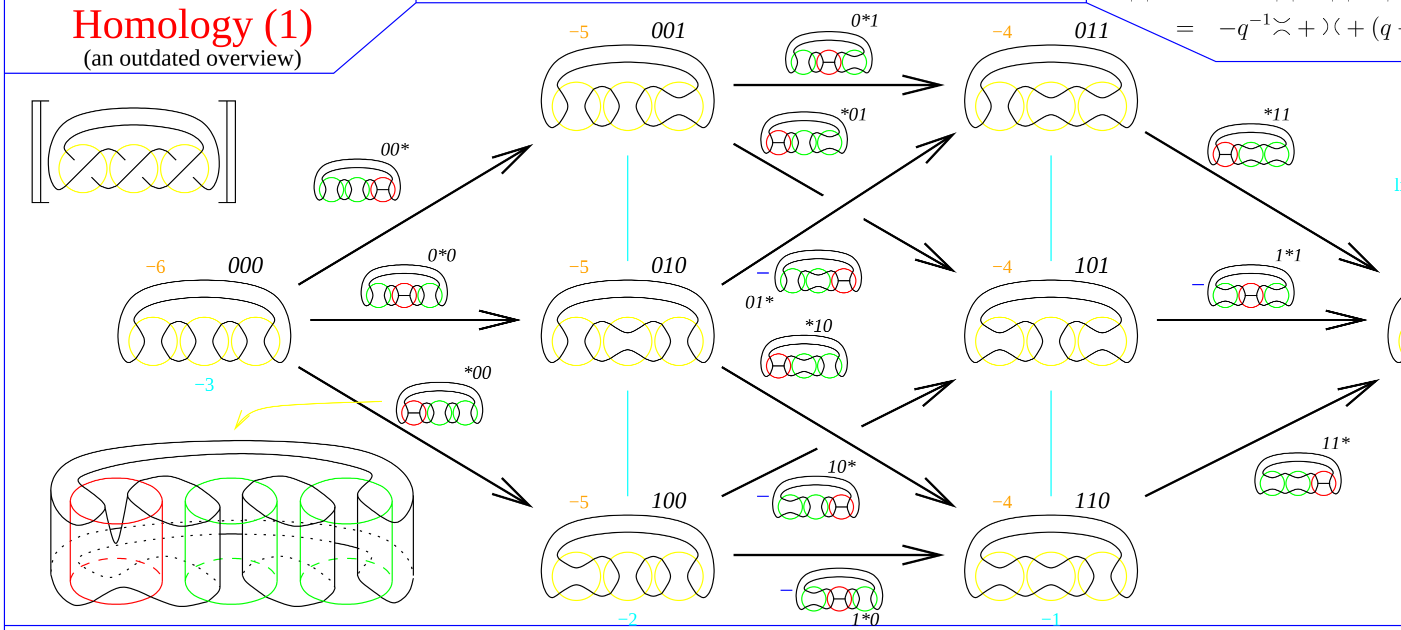
Local Khovanov Homology (1)

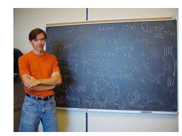
(an outdated overview)

The Jones polynomial:

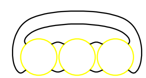
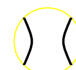
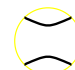


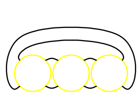
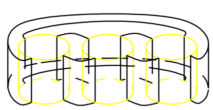
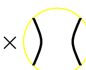
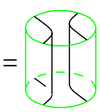
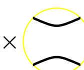
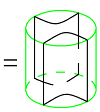
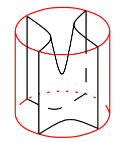




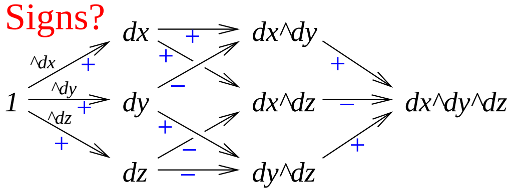

Mikhail Khovanov

What is it?

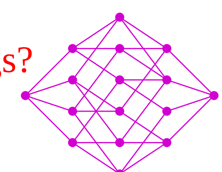
A cube for each knot/link projection; Vertices: All fillings of  with  or with  .

Edges: All fillings of $I\times$  =  with $I\times$  =  or with $I\times$  =  and precisely one  .

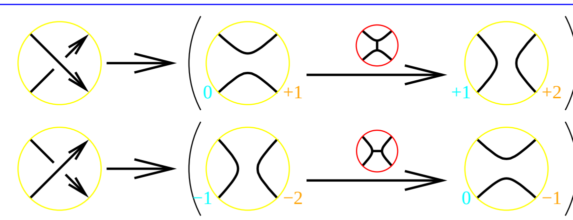
Signs?



More crossings?



General Crossings



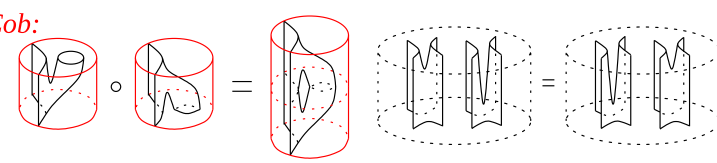
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

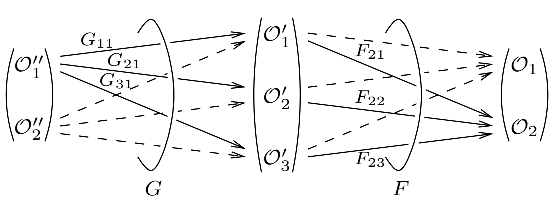
Kom: Complexes   Mat: Matrices

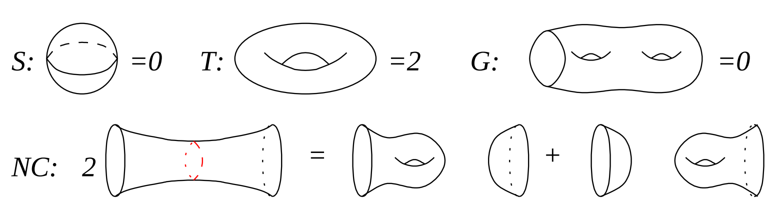
Cob: Cobordisms   <...>: Formal lin. comb.

Cob:



Mat(C):





Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+} \right)$$

Morphisms:

$$\begin{tikzcd} \cdots \arrow[r] & \Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, "F^{r-1}"'] & \Omega_0^{r} \arrow[r, "d^{r}"] \arrow[d, "F^{r}"'] & \Omega_0^{r+1} \arrow[r] \arrow[d, "F^{r+1}"'] & \cdots \\ \cdots \arrow[r] & \Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^{r} \arrow[r, "d^{r}"] & \Omega_1^{r+1} \arrow[r] & \cdots \end{tikzcd}$$

Homotopies:

$$\begin{tikzcd} \Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, shift right, "F^{r-1}"'] \arrow[d, shift left, "G^{r-1}"] & \Omega_0^{r} \arrow[r, "d^{r}"] \arrow[dl, "h^{r}"'] \arrow[d, shift right, "F^{r}"'] \arrow[d, shift left, "G^{r}"] & \Omega_0^{r+1} \arrow[dl, "h^{r+1}"'] \arrow[d, shift right, "F^{r+1}"'] \arrow[d, shift left, "G^{r+1}"] \\ \Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^{r} \arrow[r, "d^{r}"] & \Omega_1^{r+1} \end{tikzcd}$$

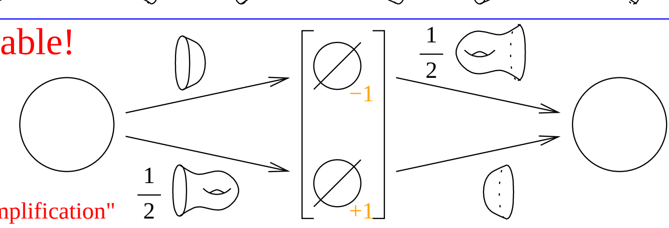
$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

The Main Point. "The cube", $\mathrm{Kh}(L)$, is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

Computable!

via "complex simplification"



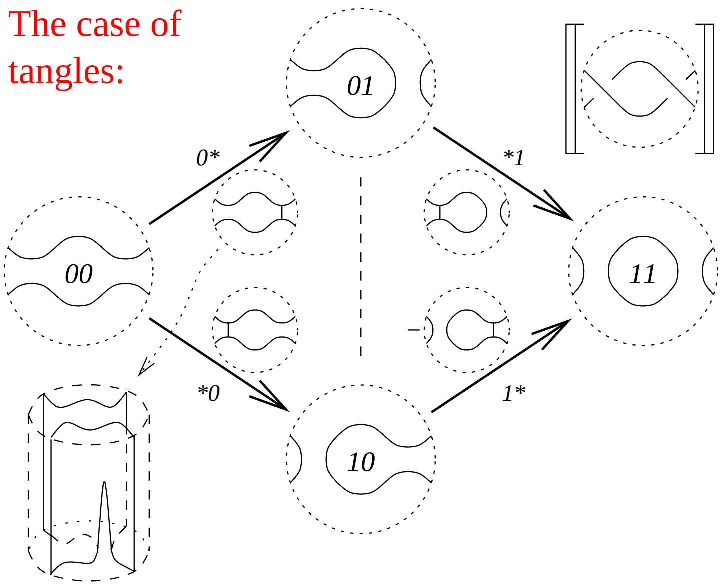
The Categorification Speculative Paradigm. • Every object in math is the Euler characteristic of a complex.

- Every operation lifts to an operation between complexes.
- Every identity remains true, up to homotopy.

Dror Bar−Natan: Talks: Montreal−1306:

Local Khovanov Homology (2)

The case of tangles:



The Reduction Lemma.

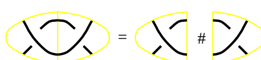
The Reduction Lemma. If ϕ is an isomorphism then the complex

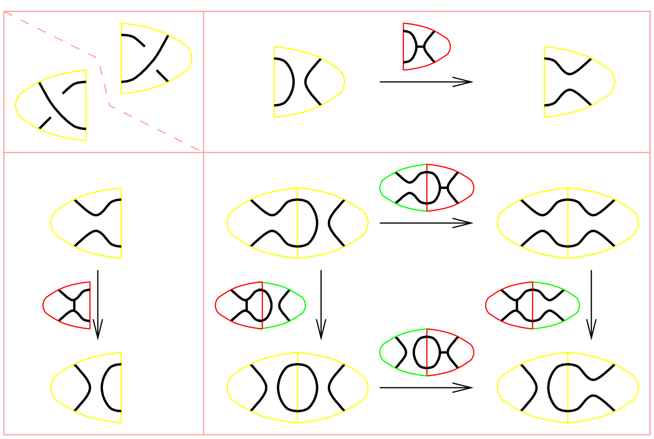
$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

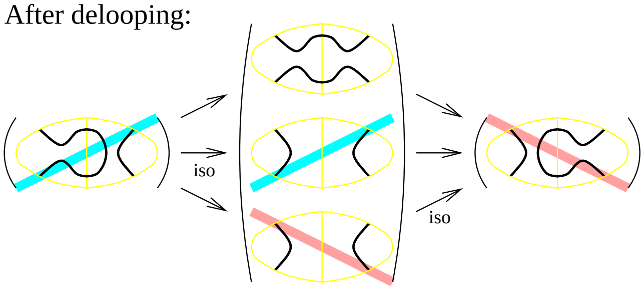
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.



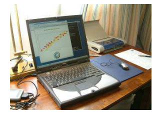


After delooping:



Kurt Reidemeister

I mean business.

In 1 day  says T(7,6)  $\dim_j H_r$ is given by:

Old techniques:

~1,000 years,
~1GGb RAM.

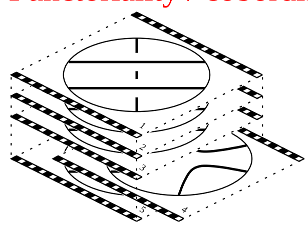
(now down to seconds)

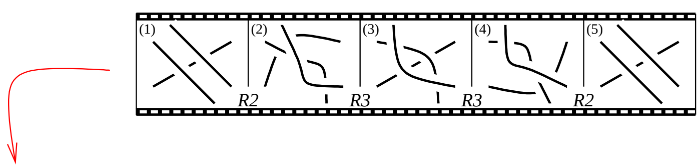
$j \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

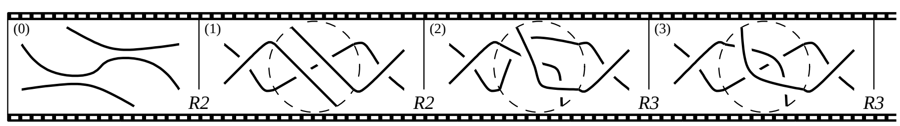
the alternating knots zone

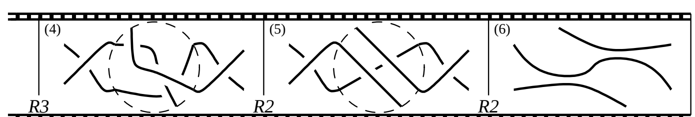
Functoriality / cobordisms.

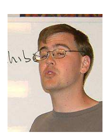
M. Jacobsson







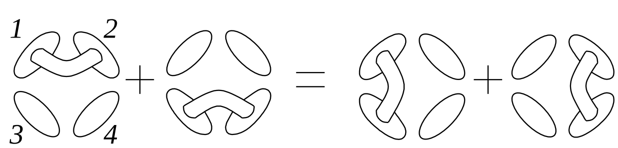




J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory:

Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots,

all else in topology is the work of mortals"

Leopold Kronecker (modified)

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Edit!

http://katlas.org

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