2019.7.11 @ Montreal "Factorization of the universal R matrix and the universal quantum invariant for framed 3-mfds"

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Contents

1. – 1984. Jones poly. ⟶ quantum invs for links.

Atiyah's question : What is 3-dim interpretation of Jones poly?

– 1989. An answer by Witten (using Chern-Simons theory)

– 1990. math def by Reshetikhin–Turaev (WRT, RT)

⟶ quantum invs for 3-mfds

A family of q-invs for framed links

– Kontsevich inv

– Universal quantum inv ($\mathcal{R}$: ribbon Hopf alg)

– RT inv ($\mathcal{R}$. fin dim reps of $\mathcal{R}$)

$\mathcal{R}=U_h(\mathfrak{sl}_2)$ └ colored Jones poly

2-dim irr rep. └ Jones poly.

$\mathcal{R}=U_h(\mathfrak{g})$

reps

Dehn surgery

for closed 3-mfds

– LMO

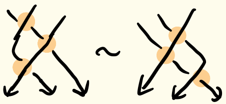
– unified WRT inv (ZHS)

– WRT inv

Key point of the construction of q. invs.

w/ diagram

 $\mapsto$ "R-matrix"

 $\mapsto$ "hexagon identity"

$\left(\text{three } Rs \right) = \left(\text{three } Rs \right)$

⟨e.g.⟩

RT : $R \in \mathrm{End}(V \otimes V)$ R-matrix

$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R) \in \mathrm{End}(V^{\otimes 3})$

Univ q inv : $\bar{R} \in \mathcal{R}^{\otimes 2}$ universal R-matrix

$\bar{R}_{12}\bar{R}_{13}\bar{R}_{23} = \bar{R}_{23}\bar{R}_{13}\bar{R}_{12} \in \mathcal{R}^{\otimes 3}$

↤ rep

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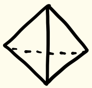
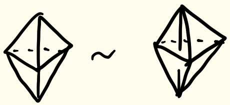
w/ triangulation

 $\mapsto$ "S-tensor"

 $\mapsto$ "Pentagon identity"

Pachner (2,3)-move

⟨e.g⟩ state-sum invs

Turaev-Viro : 6j-symbol

QHI : quantum dilog.

(Kashaev, Basailhac, Benedetti)

$\leftsquigarrow U_q(sl_2)$

Universal one ?

Framework for studying q. invs

3-mfd : compact. conn. ori.

$$\begin{array}{ccc}
\{\text{closed 3-mfd}\}/\sim & \hookrightarrow & \{\text{3-mfd}\}/\sim \\
\uparrow \text{Dehn surgery} & & \uparrow \ \text{T-V, QHI} \\
\{\text{framed link}\}/\sim & \longrightarrow & \{\text{complements}\}/\sim \\
\text{3 dim} \ \downarrow \text{proj.} \ \text{2 dim} \ \updownarrow & & \downarrow \text{3 dim} \\
\{\text{link diag}\}/\sim_R & \xrightarrow[\text{R-matrix}]{} & \text{q. invs.}
\end{array}$$

LMO. (unified) WRT

Kontsevich. Univ. colored Jones

I want to construct this path & to understand whole picture uniformly.

(s.t. $\circlearrowright$? , ...)

// 1. Intro

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2. alg. side. Drinfeld double (Drinfeld '87)

$A = (A, \eta, \mu, \varepsilon, \Delta, \gamma^{\pm 1})$: f.d. Hopf alg $/k$

$D(A) = (A^* \otimes A, \eta_{D(A)}, \mu_{D(A)}, \varepsilon_{D(A)}, \Delta_{D(A)}, \gamma_{D(A)}^{\pm 1}, R)$

: quasi-triangular Hopf alg

$R = \sum 1 \otimes e_\alpha \otimes e^\alpha \otimes 1 \in D(A)^{\otimes 2}$

$\Rightarrow R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$: 6-term eq.

Heisenberg double

$\rightsquigarrow H(A) = (A^* \otimes A, \eta_{H(A)}, \mu_{H(A)})$: alg

$S = \sum 1 \otimes e_\alpha \otimes e^\alpha \otimes 1 \in H(A)^{\otimes 2}$

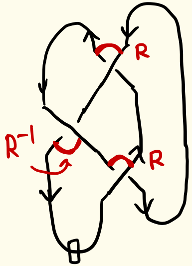
$\Rightarrow S_{12} S_{13} S_{23} = S_{23} S_{12}$: 5-term eq.

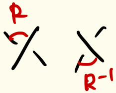
[S] arXiv:1612.08262

$\rightsquigarrow$ universal $D(A)$ inv

Sketch of the reconstruction in [S]

Original : $R = \sum \alpha \otimes \beta \;\left(= \sum 1 \otimes e_i \otimes e^i \otimes 1 \right)$

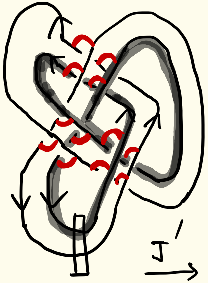


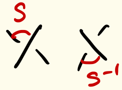


$\uparrow \; x \to \gamma_{D(A)}(x)$

$$\xrightarrow{J} \sum \bar{\alpha}\, \beta_1 \alpha_2 \bar{\beta}\, \gamma(\alpha_1)\, \gamma(\beta_2) \in D(A)$$

Reconstruction : $S = \sum \alpha \otimes \beta$





$\downarrow \; x \to \mathcal{H}(x)$

$\uparrow \; x \to \gamma(x)$

$\Uparrow \; x \to \gamma^2(x)$

$$H(x) = \begin{cases} 1 \otimes \mathcal{H}(e_i) & : x = \alpha \\ \gamma^{-1}(e^i) \otimes 1 & : x = \beta \end{cases}$$

$$\xrightarrow{J'} \sum \overline{\alpha_{j2}}\, \overline{\alpha_{i2}}\, \beta_{j1} \beta_{i1} \alpha_{l3} \alpha_{j3} \overline{\beta_{k2}}\, \overline{\beta_{j2}} \cdots$$
$$\otimes\; \gamma(\overline{\alpha_{k2}})\, \gamma(\overline{\alpha_{l2}})\, \mathcal{H}(\beta_{k1})\, \gamma(\beta_{l1}) \cdots \in H(A) \otimes H(A)^{op}$$

Thm (Kashaev '95)

$$\exists\, \varphi : D(B) \overset{alg}{\hookrightarrow} H(B) \otimes H(B)^{op}$$

$$\varphi^{\otimes 2}(\check{R}) = S^*_{14} S^*_{13} S^*_{24} S^*_{23} \quad //$$

$S^*_{ij} = \sum \gamma^*_i(\alpha) \otimes \gamma^*_j(\beta)$, $\gamma^*_i = \begin{cases} id, & i : \text{odd} \\ \gamma, & i : \text{even} \end{cases}$

Thm ([S] '18)

L : n-comp link w/ base pt

$$J'(L) = \varphi^{\otimes n}(J(L))$$

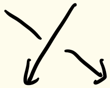
// 2. alg side

3. geom. side.

Octahedral decomposition (Yokota, Cho, Kim, Kim)

ideal triangulation in $S^3 \setminus \{\pm\infty\}$

diagram



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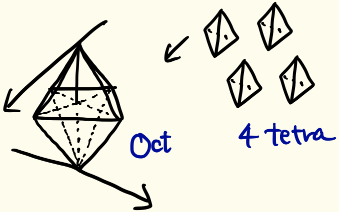
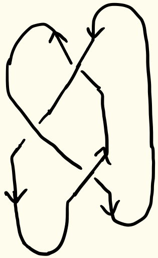


⟨e.g.⟩ trefoil



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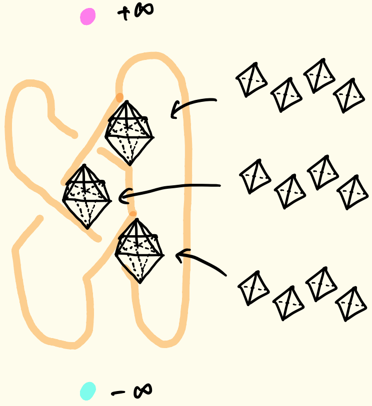

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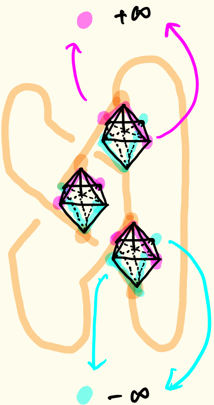

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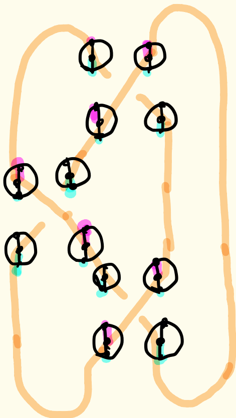


boundary of oct. ··· 

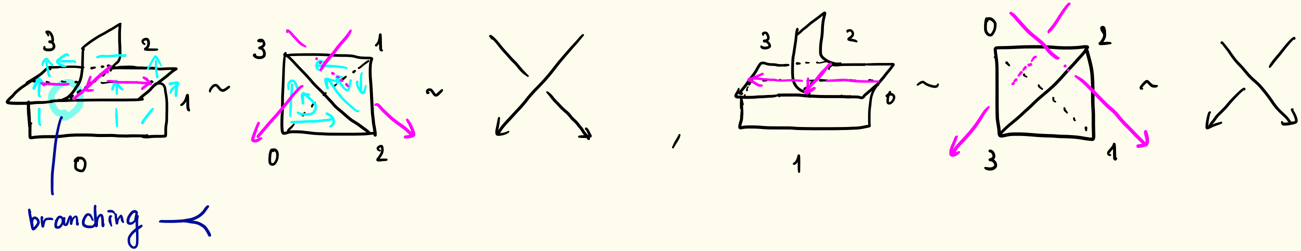
↝ paste along strings

↝ ideal triangulation of

$S^3 \setminus ($$\cup \{\pm\infty\})$

branched spine (Benedetti – Petronio '97)

{virtual tangle diagram} ~ {"N-graph" for branched spine}

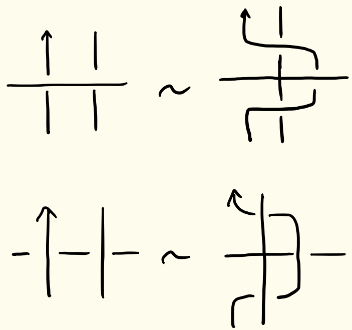


$\{\text{closed 3-mfds}\}/\sim \;\longleftrightarrow\; \{\text{"nice" N-graph}\}/P_{closed} = \mathcal{M}$

$\{\;\text{" w/ combing}\}/\sim \;\longleftrightarrow\; \{\;\text{"}\;\}/P_{comb} = \mathcal{M}_c$

$\{\;\text{" w/ framing}\}/\sim \;\longleftrightarrow\; \{\;\text{" w/ } \mathbb{Z}/2\mathbb{Z} \text{ coloring}\}/P_f = \mathcal{M}_f$

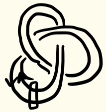
e.g. in $\mathcal{M}_c$ (so in $\mathcal{M}$)



(Pachner (2,3))

What we did in the reconstruction:

$\{\text{link diagram w/ b.p}\} \longrightarrow \{N\text{-graph w/ b.p}\}$

 $\longmapsto$  : octahedral decomposition of $S^3 \setminus (\ \cup \{\pm\infty\})$

+ branching.

$J \downarrow$ $\qquad$ $J' \downarrow$

$$D(A) \overset{\varphi}{\hookrightarrow} H(A) \otimes H(A)^{op}$$

Q What is J' on $\{N\text{-graph}\}$?

(What relations J' is inv on them?)

> <u>Thm</u> (Terashima – S)
>
> If A is involutory, then
>
> $$J' : \mathcal{M} \longrightarrow H(A) \otimes H(A)^{op}$$

For general A, <u>$\mathcal{M}$ is too strong</u>!

($\longleftrightarrow$ J' could catch geometry)

v.s. Kuperberg inv ?

v.s. Turaev-Viro inv ?

v.s. QHI ?

// 3. geom. side

4. Summary

$$
\begin{tikzcd}
\{\text{closed 3-mfd}\}/\sim \arrow[rr, hook] & & \{\text{3-mfd}\}/\sim & \\
\{\text{framed link}\}/\sim \arrow[u, "\text{Dehn surgery}"] \arrow[rr] \arrow[d, leftrightarrow, "\text{3 dim} \xrightarrow{\text{proj.}} \text{2 dim}"'] \arrow[rrd, "\text{LMO. (unified) WRT}"] \arrow[rrd, "\text{Kontsevich. Univ. colored Jones}"'] & & \{\text{complements}\}/\sim \arrow[u, hook, "\text{T-V, QHI}"'] \arrow[d, "\text{3 dim}"] & \\
\{\text{link diag}\}/\sim_R \arrow[rr, squiggly, "\text{R-matrix}"'] & & \text{q. invs.} &
\end{tikzcd}
$$

+geometry

I want to construct this path & to understand whole picture uniformly.
(s.t. , ?)

Future

- Extend the universal quantum inv to an inv of (framed?) 3-mfd
- Find similar framework for Kontsevich inv. (w/ associator ?)
- Compare them to T-V, QHT, LMO
- Quantum theory for Heisenberg double ?

// end