MAT240

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$W \subseteq V$ is a "subspace" if

a) It contains 0 and is closed under addition & multiplication by scalars

$\Leftrightarrow$

b) In itself, it is a vectorspace with the operations needed from V.

$x, y \in W \Rightarrow x+y \in W$

$c \in F,\ x \in W \Rightarrow cx \in W$

Def. Let $(u_i) = (u_1, u_2, \dots, u_n)$ be a sequence of vectors in V.
A sum of the form $\sum_{i=1}^{n} a_i u_i$, $a_i \in F$

$$= a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

is called a "linear combination" of the u_i

span (u_i) := the set of all possible linear combinations of the u_i's.

If $S \subset V$ is any subset

span S := the set of all linear combinations of vectors in S.

$$= \left\{ \sum_{i=0}^{n} a_i u_i \ : \ \begin{matrix} a_i \in F \\ u_i \in S \end{matrix} \right\}$$

↳ this always contains 0. even if S is empty because product of empty set $= 1$ and sum of empty set $= 0$.

Aside: $\sum_{i=1}^{0} x_i = 0 \Rightarrow$ sum of empty set $= 0$.

Thm. For any $S \in V$, span S is a subspace of V.

Pf. 1. $0 \in$ Span S. ✓

2. $x \in$ Span S, $y \in$ Span S $\longrightarrow$ $x = \sum a_i u_i,\ u_i \in S$; $y = \sum b_i v_i\ \ v_i \in S$.

$$x+y = \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n+m} c_i w_i$$

$$(c_i) = (a_1 \dots a_n b_1 \dots b_m)$$

$$(w_i) = (u_1 \dots u_n v_1 \dots v_m).$$

3. $x \in$ Span S, $x = \sum a_i u_i$, $u_i \in S$.

$$cx = c\sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{n} (ca) u_i \in \text{Span } S.$$

Ex. Let $P_3(\mathbb{R}) = \{ax^3+bx^2+cx+d\} \subset P(\mathbb{R})$, $a, b, c, d \in \mathbb{R}$ ($P_3(\mathbb{R}) \to V$)

$u_1 = x^3 - 2x^2 - 5x - 3$
$u_2 = 3x^3 - 5x^2 - 4x - 9$
$v = 2x^3 - 2x^2 + 12x - 6$

Let $W =$ span(u_1, u_2)

Question: Does $v \in W$?

v is in W if

$$v = a_1 u_1 + a_2 u_2$$

for some a_1 & $a_2 \in \mathbb{R}$.

If $\exists a_1, a_2 \in \mathbb{R}$ $2x^3 - 2x^2 + 12x - 6 = a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9)$

$$= (a_1 + 3a_2)x^3 + (-2a_1 - 5a_2)x^2 + (-5a_1 - 4a_2)x + (-3a_1 - 9a_2)$$

Solve system

$$\Longleftrightarrow \begin{cases} 2 = a_1 + 3a_2 \\ -2 = -2a_1 - 5a_2 \\ 12 = -5a_1 - 4a_2 \\ -6 = -3a_1 - 9a_2 \end{cases}$$

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If we find a_1 & a_2 which satisfy equ'ns, then $V \in W$.

Suppose: ① $2 = a_1 + 3a_2$ $\xrightarrow{\cdot 2}$ ② $4 = 2a_1 + 6a_2$

$-2 = -2a_1 - 5a_2$ $\xrightarrow{\cdot 1}$ $-2 = -2a_1 - 5a_2$

$\Downarrow$ $\boxed{2 = a_2}$

$0 = -a_1 - 2a_2$.

③ $\boxed{a_1 = -4}$

Now, check that it really holds for 4 original equ'ns.
If do $\Rightarrow$ V is indeed in W.