MAT401 April 9 2008

Today

1. On the final
2. Fix a blunder
3. State lemmas 2 & 4.
4. Proof of $\psi \circ \Phi = I$ $\quad E\,Gal(E/K) = K$
5. Proofs of lemmas 2 & 4

Thm For a splitting extension E/F:

$$\begin{array}{lcll}
E & \longleftarrow & \{e\} = Gal(E/E) & \\
\big| [E:K] & & \quad |H| \big| & \\
K & \longleftarrow & H = Gal(E/K) & \text{If } K \text{ is splitting, } H \triangleleft G \\
\big| [K:F] & & \quad \big| [G:H] & \text{and } Gal(K/F) = G/H \\
 & & & = Gal(E/F)/Gal(E/K) \\
F & \longleftarrow & G = Gal(E/F) &
\end{array}$$

Suppose f splits over $F(a_1, \ldots, a_n)$
s.t. $b_i = a_i^{n_i} \in F(a_1 \ldots a_{i-1})$
for each.
Then $Gal(S_f(F)/F)$ is solvable.

$$\begin{array}{lll}
F(a_1 \ldots a_n) \subset E_n = & & K_n = S_{E_n}(E) \\
F(a_1 \ldots a_{n-1}) \subset E_{n-1} & & K_{n-1} = S_{E_{n-1}}(F_1) \\
\quad | \qquad\qquad | & & \qquad | \\
F(a_1 a_2) \subset E_2 = S_E(x^{n_2} - b_2) & & K_2 = S_{E_2}(F) \\
\quad | \qquad\qquad | & & \qquad | \\
F(a_1) \subset E_1 = S_E(x^{n_1} - b_1) & & K_1 = S_{E_1}(F) \\
\quad | \qquad\qquad | & & \qquad | \\
F \quad = \quad E_0 & \text{———} & K_0 = S_{E_0}(F)
\end{array}$$

Example

($\sqrt[4]{2}, i\sqrt{2}$ $-\sqrt{2}, -i\sqrt{2}$)

$x^4 - 2 = 0$

non splitting

$$\begin{array}{l} \mathbb{Q}(\sqrt[4]{2}) \quad x^2 - \sqrt{2} = 0 \quad (x-\sqrt{2})(x+\sqrt{2}) \\ \;2| \qquad\qquad x^2 = 2 \\ \mathbb{Q}(\sqrt{2}) \quad \overset{\shortparallel}{(x-\sqrt{2})}(x+\sqrt{2}) \\ \;2| \\ \mathbb{Q} \end{array}$$

claim: For every i, $Gal(k_i/E_i)$ is solvable.

Proof: for $i=n$, $Gal(k_n/E_n) = \{e\}$

Assume $Gal(k_{i+1}/E_{i+1})$ is solvable.

$$\begin{array}{lll} & \text{solvable} & k_{i+1} = S_{E_{i+1}}(f) \\ E_{i+1} = S_{E_i}(x^{n_{i+1}} - b_{i+1}) & & \\ \text{Solvable} \; | & \text{splitting field} & k_i = S_{E_i}(f) \\ E_i & \text{Solvable} & \end{array}$$

$G = Gal(k_{i+1}/E_i)$

$H = Gal(k_{i+1}/E_{i+1})$ (Solvable by induction)

$G/H \underset{\text{Thm}}{\overset{\text{by End}}{=}} Gal(E_{i+1}/E_i)$

Solvable by lemma

$\Rightarrow G$ is solvable $\Rightarrow$

$Gal(k_i/E_i)$, a quotient of G, is also solvable.

Thm For a splitting extension E/F.

$\{k : E/k/F\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{H : H < G = Gal(E/F)\}$.

$\Phi(k) = Gal(E/k)$

$\Psi(H) = E_H$

claim $\psi \circ \Phi = I$. ie, if

$$k \xmapsto{\Phi} Gal(E/k) \xrightarrow{\psi} E_{Gal(E/k)} = k.$$

lemma 2. (uniqueness of splitting fields)

Suppose $\phi: F_1 \to F_2$ is an isomo of fields, $f_1 \in F_1(x)$, $f_2 = \phi(f_1) \in F_2[x]$.
Suppose E_i is a splitting field for f_i over F_i for $i = 1, 2$. Then $\exists$ iso $\tilde{\phi}: E_1 \to E_2$ s.t $\tilde{\phi}|_F = \phi$

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{\phi}"', "\exists"] \arrow[d, dash] & E_2 \arrow[d, dash] \\ F_1 \text{ in } F_1 \arrow[r, "\phi"] & F_2 \quad \text{"}\ni\text{"}\ F_2 \end{tikzcd}$$

lemma 4. "splitting field's split well".

Suppose E/F is a splitting ext, & $P \in F[x]$ is irreducible, then if E contains a root of p, then p splits in E.

Proof of $\psi \cdot \Phi = I$:

$k \subset E_{Gal(E/k)}$ trivial
$k \supset E_{Gal(E/k)}$: assume $w \in E$. but $w \notin k$.
$| |^{c} 1$.

$$\begin{tikzcd} & E & \\ k(w) \arrow[rr, "\phi"] \arrow[dr, dash] & & k(w) \arrow[dl, dash] \\ & k. & \end{tikzcd}$$

let p be the minimal poly of w over k.

($p(\omega)=0$; & p is irred). by lemma 4, p splits in E $\deg p > 1$
(if $\deg p = 1 \Rightarrow \omega \in k \Rightarrow\Leftarrow$).

So p has another root $u \in E$. Then $k(\omega) \cong k[x]/\langle p\rangle \cong k(u)$
so $\exists$ an iso $\phi : k[\omega] \to k(u)$, s.t $\phi(\omega) = u$

If E is $S_k(f)$, then $E = S_{k(\omega)}(f) = S_{k(u)}(f)$,

by lemma 2, $\exists$ iso $\tilde{\phi} : E \to E$
s.t $\tilde{\phi}|_{k(\omega)} = \phi$ so $\tilde{\phi} \in Gal(E/k)$
yet $\tilde{\phi}(\omega) = u$ so $\omega \notin E^{Gal(E/k)}$ □

Proof of lemma 2 by induction

on $\deg f_1 = \deg f_2$

If $\deg = 1$, there is nothing to prove.

$E_1 = F_1$; $E_2 = F_2$, so take $\tilde{\phi} = \phi$

Assume then holds for $\deg f_1 < 7$, assume $\deg f_1 = 7$.

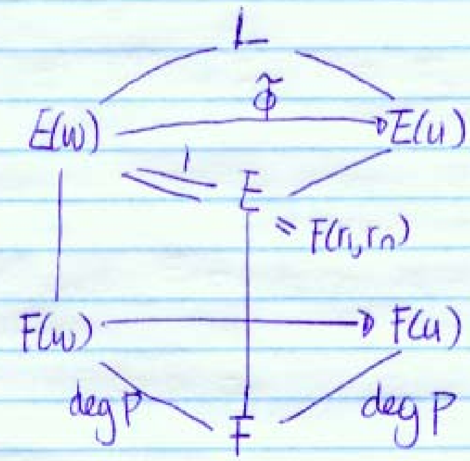
$$\begin{array}{ccccc}
\text{a split ext of } f_1/x-\omega_1 \left\{ \right. & E_1 & \xrightarrow{\tilde{\phi}} & E_2 & \text{a split ext } E/x-\omega_2 \\
 & | & & | & \\
 & F(\omega_1) & \xrightarrow{\bar{\phi}} & F(\omega_2) & \\
 & | & & | & \\
 & F_1 & \xrightarrow{\phi} & F_2 & \\
 & \downarrow & & \downarrow & \\
 & p_1 | f_1 & \xrightarrow{\phi} & f_2\, p_2 &
\end{array}$$

let P_1 be some irreducible factor of f_1 & let $P_2 = \phi(P_1)$
then P_2 is an irred factor of f_2.

let w_i be a root of P_i in E_i for $i = 1, 2$.
then
$$F_1(w_1) \cong F_1[x]/\langle P_1 \rangle \xrightarrow{\phi} F_2[x]/\langle P_2 \rangle \cong F_2(w_2)$$

So $F_1(w_1) \cong F_2(w_2)$.

with some $\tilde{\phi}$
by induction $\exists\, \tilde{\phi} : E_1 \to E_2$.



$= F(u, r_1 \ldots r_n)$
$= F(u)(r_1, \ldots r_n)$

So $[E(w):F] = [E(u):F]$
so $[E(w):E][E:F] = [E(u):E][E:F]$
$\Rightarrow [E(u):E] = 1$