

Homework 9. NOT HAND IN.

Section 13.

1. a) $\int_S f - \int_S g = \int_S (f-g)$ make sense since f, g are integral over S .
 (can use the THM 11.3 (1). Let P be an partition over S . $L(f-g, P) = \sum_{R \in P} V(R) m_R(f-g)$.

"if f vanishes except on a set of measure-0, then $\int f = 0$ ".

(a) let A be the set of measure-0, s.t. ACS.
 then $L(f-g, P) = \sum_{R \in P, R \subset A} V(R)m_R(f-g) + \sum_{R \in P, R \cap A \neq \emptyset} V(R)m_R(f-g) \leq \max[m_R(f-g)] \sum_{R \in P, R \cap A \neq \emptyset} V(R) + 0 < \epsilon + 0$,
 since A has measure-0, so \exists V_R covers A . s.t. $\sum V_R < \frac{\epsilon}{\max[m_R(f-g)]}$
 Since $f=g$ on $S \setminus A$. so $m_R(f-g) = 0$ on $S \setminus A$.

$$\text{so } \int_S (f-g) = \sup_P L(f-g, P) = 0 \text{ since as } \epsilon \rightarrow 0, L(f-g, P) \rightarrow 0.$$

$$\text{Hence, } \int_S f = \int_S g.$$

b) since $\int_S f = \int_S g$, so $\int_S g - \int_S f = \int_S g - f = 0$. & $g-f$ is integrable over S .
 since $f(x) \leq g(x)$, so $g(x) - f(x) \geq 0$ for $x \in S$.

By THM 11.3 (2) "if f is non-negative & if $\int f = 0$, then f vanishes except on a set of measure-0".

we can get $g-f=0$ except on a set of measure-0. as required.

2. By Fubini THM, since $\int_S f$ exists, both $\int_{y \in B} f(x, y)$ & $\int_{x \in A} f(x, y)$ are integrable.
 & $\int_S f = \int_A \bar{I}(x) = \int_A I(x)$ & $\bar{I}(x) \geq I(x)$ for all $x \in A$.

we know $\int_{y \in B} f(x, y)$ exists iff $I(x) = \bar{I}(x)$

Setting $g(x) = \bar{I}(x) - I(x)$. so $g(x) \geq 0$ for all $x \in A$, that g is integrable over A &
 $\int_A g = \int_A (\bar{I} - I) = \int_A \bar{I} - \int_A I = 0$.

Then by THM 11.3 (2), since $g \geq 0$ on A & $\int_S g = 0$ then $g = 0$ except on a set of measure-0.

So $\bar{I}(x) - I(x) = g(x) = 0$ when $x \in A - D$, where D is a set of measure-0.
 Hence $\int_{y \in B} f(x, y)$ exists for $x \in A - D$.

3. base case = $b=2$. prove $\int_S f = \int_{S_1} f + \int_{S_2} f$ where $S_1 \cup S_2 = S$, & $S_1 \cap S_2$ is measure-0.
 by THM, $\int_S f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$.

since $\int_{S_1 \cap S_2} f = \int_{S_1 \cap S_2} g$ since $f_{S_1 \cap S_2}$ vanishes except on $S_1 \cap S_2$ has measure-0.

then by THM 11.3 (1). $\int_{S_1 \cap S_2} f = 0$.

so $\int_S f = \int_{S_1} f + \int_{S_2} f$. as required.

inductive step: $R=n$. s.t. $\int_S f = \int_{S_1} f + \dots + \int_{S_n} f$.

want to prove $R=n+1$, $S=S_1 \cup \dots \cup S_{n+1}$, $S \cap S_j$ has measure-0. when it's.

then $\int_S f = \int_{S_1} f + \dots + \int_{S_{n+1}} f$.

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for $S' = S_1 \cup \dots \cup S_n$, s.t. $S_i \cap S_j$ has measure-0 when $i \neq j$.

Then by induction hypothesis $\int_{S'} f = \int_S f + \dots + \int_{S_n} f$.

Since $S = S' \cup S_{n+1}$ & $S' \cap S_{n+1} = (S_1 \cap S_{n+1}) \cup \dots \cup (S_n \cap S_{n+1})$ has measure-0.

Since $S_1 \cap S_{n+1}, \dots, S_n \cap S_{n+1}$ has measure-0, so their union also has measure-0.
Then $\int_S f = \int_{S'} f + \int_{S_{n+1}} f = \int_{S_1} f + \dots + \int_{S_n} f + \int_{S_{n+1}} f$. as required. \square

5. Let $S = Q \cap [0,1]$, $f = 1$ S is bounded. & $A = \text{Int } S$ = a set of measure-0.

so $\int_A 1 = 0$ exists. but $\int_S 1 = \int_Q 1_S = \int_Q 1_{S \setminus A}$ is not integral on Q ,
 $x \notin S$. i.e. not exists.

6. ^{"Reversion"} THM 13.6. Let S be a bounded set in \mathbb{R}^n ; let $f: S \rightarrow \mathbb{R}$ be bounded function.

Let $A = \text{Int } S$. If f is integrable over S , then f is integrable over A .

$$\& \int_S f = \int_A f$$

Proof: Since f is integrable over S , then by definition, (discont. set)^P of f has measure-0. in S . so. discont-set of f has measure-0 in A as well $\Rightarrow \int_A f$ exists.

So. f is cont. on $S \setminus D = S'$, & let $A' = \text{Int } S'$, & f is integrable over S' .

So by THM 13.6. f is integrable over A' , & $\int_{S'} f = \int_{A'} f$.

$$\text{Since } \int_S f = \int_{S \setminus D} f = \int_{S'} f - \int_D f. = \int_{S'} f - \int_A f = \int_A f - 0.$$

since f_D vanishes except on D (set of measure-0).

$$\text{so } \int_S f = \int_{S'} f = \int_{A'} f. \Rightarrow \int_A f = \int_S f.$$

$$\text{Since } A' \subset A \Rightarrow \int_{A'} f \leq \int_A f. \quad \Rightarrow \quad \int_A f = \int_S f.$$

$$\text{Since } A \subset S \Rightarrow \int_A f \leq \int_S f.$$

Section 14.

1. (a) Since S_1, S_2, \dots, S_n are rectifiable sets,

By THM 14.2 (C) (Additivity), if S_1, S_2 are rectifiable, so does $S_1 \cup S_2$ is rectifiable.

Repeat applying the THM, we can get $S_1 \cup S_2 \cup \dots \cup S_n$ is rectifiable.

(b)

2. Since S_1, S_2 are rectifiable, then $Bd(S_1)$ & $Bd(S_2)$ have measure-0.

Since $Bd(S_1 \cup S_2) = Bd(S_1') \cup Bd(S_2')$ where $Bd(S_1') \subseteq Bd(S_1)$ & $Bd(S_2') \subseteq Bd(S_2)$

so $Bd(S_1')$ & $Bd(S_2')$ have measure-0. then their union has meas-0.

so $Bd(S_1 \cup S_2)$ has measure-0. since $S_1 \cup S_2$ is bounded by S .

Then $S_1 \cup S_2$ is rectifiable.

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$$V(S_1 \cup S_2) = \int_{S_1 \cup S_2} I = \int_Q I_{S_1 \cup S_2} = \begin{cases} 1 & x \in S_1, x \notin S_2 \\ 0 & \text{otherwise.} \end{cases}$$

$$V(S_1) - V(S_1 \cap S_2) = \int_{S_1} I - \int_{S_1 \cap S_2} I = \int_Q I_{S_1} - \int_Q I_{S_1 \cap S_2} = \begin{cases} 1-0=1 & x \in S_1, x \notin S_2 \\ 1-1=0 & x \in S_1, x \in S_2 \\ 0-0=0 & x \notin S_1, x \notin S_2 \\ 0-0=0 & x \notin S_1, x \in S_2 \end{cases} = \begin{cases} 1 & x \in S_1, x \notin S_2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence. $V(S_1 \cup S_2) = V(S_1) - V(S_1 \cap S_2)$

3. Since A is a nonempty, rectifiable open set in \mathbb{R}^n .

so. $\int_A I = \int_Q I_A$ exists. $= \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

Since A is not empty, we can find an element $x \in A$.

Since A is open. So $\exists \epsilon > 0$, s.t. $V(x, \epsilon) \subset A$. & $V(x, \epsilon)$ is rectifiable.

then $V(A) \geq V(V(x, \epsilon)) > 0$ since $\epsilon > 0$

5.

6. Let $A = [0, 1] \cap \mathbb{Q}^n$ $f(x) = 1$. bounded & cont.

then $\bar{A} = [0, 1]^n$ $\int_{\bar{A}} f = \int_{[0, 1]^n} I = 1^n = 1$

but $\int_A f = \int_A I$ does not exist since A is not rectifiable $\Leftrightarrow \text{Bd}(A) = [0, 1]^n$ not has measure-0

7. a). Since S is rectifiable, then $\text{Bd } S$ has measure-0. & S is bounded
so \bar{S} is bounded, & claim: $\text{Bd}(\bar{S}) \subset \text{Bd}(S)$

Proof claim: Let $x \in \text{Bd}(\bar{S})$, want to show $x \in \text{Bd}(S)$. (i.e.) want to show. V of x contains points of S & of $\mathbb{R}^n \setminus S$.

So let V be a nbd of x . Since $x \in \text{Bd}(\bar{S})$, V contains point not in \bar{S} .
s.t. those points not in S .

Being in \bar{S} , every nbd of x intersects S , thus V contains points in S .
Hence. $x \in \text{Bd}(S) \Rightarrow \text{Bd}(\bar{S}) \subset \text{Bd}(S)$

So. $\text{Bd}(\bar{S})$ has measure-0, then by the definition, \bar{S} is rectifiable.
 $V(S) \leq V(\bar{S}) = V(S) + V(\text{Bd}(S)) - V(S \cap \text{Bd}(S)) = V(S)$

so $V(S) = V(\bar{S})$ as required.

b) Let $S = \mathbb{Q} \cap [0, 1]$ in \mathbb{R} . S is not rectifiable since $\text{Bd}(S) = [0, 1]$ not has measure-0
then $\bar{S} = [0, 1]$. & $\text{Int } S = \emptyset$. are rectifiable.