

Prof: Dror Bar-Natan.

Date: Apr 4/07

HM #9.

(2) Show that the dihedral groups are solvable.

$$D_n = \langle r, s \mid r^n = e, s^2 = e, srs^{-1} = r^{-1} \rangle.$$

$$\text{Let } N = \langle r \rangle.$$

$$= \{e, r, r^2, \dots, r^{n-1}\}.$$

Let's look at

$$\{e\} = H_0 \subset H_1 = N.$$

(A) $H_0 \triangleleft H_1$ since H_0 is the trivial subgroup

$$\forall g \in H_1 \quad \forall e \in H_0 \Rightarrow geg^{-1} = g^{-1}g = e \in H_0.$$

$$(B) H_1/H_0 = N/H_0 = \{nH_0 \mid n \in N\}.$$

$$= \{ne \mid n \in N\} \quad \therefore H_0 = \{e\}$$

$$= \{n \mid n \in N\}$$

$$= N.$$

$N = \langle r \rangle$ and is cyclic because generated by one element
so it is also Abelian since it commutes.

N abelian $\Rightarrow N/H_0$ Abelian.

$\therefore N$ is solvable because of (A) and (B)

Now consider D_n/N

$$D_n/N = D_n/\langle r \rangle. \quad \{N, sN\} \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 \text{ abelian.}$$

so

D_n/N is abelian $\Rightarrow D_n/N$ is solvable.

Since N is solvable and D_n/N is also solvable.

so by theorem 32.4

(perhaps) implies that D_n is solvable.

24) Show that S_n is solvable when $n \leq 4$.

$$S_1 = \{e\}$$

$$S_2 \cong \mathbb{Z}/2\mathbb{Z}, \text{ which is abelian.}$$

$$S_3 \cong D_3$$

and D_3 is a dihedral group and by exercise 22
all dihedral groups are solvable.

Theorem:

N is normal subgroup of G , N and G/N are solvable, then G is solv.

$A_n \triangleleft S_n$ in general.

$$\text{since } |S_n| = n!$$

$$|A_n| = n!/2$$

$$|S_n|/|A_n| = 2$$

So now S_4 :

check if S_4/A_4 is abelian.

so check that A_4 is solvable.

$$|A_4| = 12.$$

choose $N_1 < A_4$ such that N_1 is solvable.

$$\text{choose } N_1 = \{e, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

N_1 is normal to A_4 .

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

A_4/N_1 is abelian since $\frac{|A_4|}{|N_1|} = \frac{12}{4} = 3$.

and $|\mathbb{Z}/3\mathbb{Z}| = 3$ and abelian.

$$\text{so } A_4/N_1 \cong \mathbb{Z}/3\mathbb{Z} \text{ and is solvable.}$$

so by theorem above A_4 is solvable.

so using the theorem again, we have.

S_4 is solvable.

(26) Show that a subgroup of a solvable group is solvable

Assume $H \subset G$

Let $H_i \cap H$ this is a subgroup of H $i=0, \dots, n-1$
and $H_i \cap H = N_i$ is a series of subgroups of H .
using the definition solvable group.
and every $N_i \triangleleft N_{i+1} \Rightarrow H_i \triangleleft H_{i+1}$.

$$\frac{N_{i+1}}{N_i} = \frac{H \cap H_{i+1}}{H \cap H_i}$$

H_{i+1}/H_i abelian: $\forall a, b \in N_{i+1} \Rightarrow aN_i b N_i = b N_i a N_i$

$$aN_1 \cdot bN_2 = aN_1 \bar{a}^1 \bar{a}^1 bN_2 \quad \text{since } \bar{a}^1 a = e.$$

let $n_3 = aN_1 \bar{a}^1$ and $n_3 \in N_i$

so we have $= n_3 b N_2$ since $b\bar{b}^{-1} = e$. & $a\bar{a} = b\bar{b}$
we have $= b\bar{b}^{-1} n_3 b a n_4$. $n_4 \in N$

let $n_5 = b\bar{b}^{-1} n_3 b$.

so we have

$$= b n_5 a n_4 \in b N_i a N_i$$

so H_{i+1}/H_i is abelian.

$\therefore \{e\} = H_0 \subset \dots \subset H_n = H$ is a solvable series. for H .

$\therefore H$ is arbitrary, any subgroup of a
solvable group is solvable