Problem Set 12 — MAT257

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Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §23 (p.202)

1. Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be the map $\alpha(x) = (x, x^2)$; let M be the image set of α. Show that M is a 1-manifold in $\mathbb{R}^2$ covered by the single coordinate patch α.
2. Let $\beta : \mathbb{H}^1 \to \mathbb{R}^2$ be the map $\beta(x) = (x, x^2)$; let N be the image set of β. Show that N is a 1-manifold in $\mathbb{R}^2$.

$*$ 3. (a) Show that the unit circle S^1 is a 1-manifold in $\mathbb{R}^2$.

(b) Show that the function $\alpha : [0, 1) \to S^1$ given by

$$\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$$

is not a coordinate patch on S^1.

4. Let $A \subset \mathbb{R}^k$ be open; let $f : A \to \mathbb{R}$ be of class $\mathcal{C}^r$. Show that the graph of f is a k-manifold in $\mathbb{R}^{k+1}$.
5. Show that if M is a k-manifold without boundary in $\mathbb{R}^m$, and if N is an l-manifold in $\mathbb{R}^n$, then $M \times N$ is a $k + l$ manifold in $\mathbb{R}^{m+n}$.

$*$ 6. (a) Show that $I = [0, 1]$ is a 1-manifold in $\mathbb{R}^1$.

(b) Is $I \times I$ a 2-manifold in $\mathbb{R}^2$? Justify your answer.

2 Munkres §24 (pp.208–209)

1. Show that the solid torus is a 3-manifold, and its boundary is the torus T. (See the exercises of §17.)

   Hint: Write the equation for T in cartesian coordinates and apply Theorem 24.4.
2. Prove the following:

   Theorem. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be $\mathcal{C}^r$. Let M be the set of all $\mathbf{x}$ such that $f(\mathbf{x}) = \mathbf{0}$. Assume that M is non-empty and that $DF(\mathbf{x})$ has rank n for $\mathbf{x} \in M$. Then M is a k-manifold without boundary in $\mathbb{R}^{n+k}$. Furthermore, if N is the set of all $\mathbf{x}$ for which

$$f_1(\mathbf{x}) = \cdots = f_{n-1}(\mathbf{x}) = 0,$$
$$f_n(\mathbf{x}) \geq 0,$$

and if the matrix

$$\partial(f_1, \ldots, f_{n-1})/\partial \mathbf{x}$$

has rank $n-1$ at each point of N, then N is a $k+1$ manifold, and $\partial N = M$

Hint: Examine the proof of the implicit function theorem.

∗ 3. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be $\mathcal{C}^r$. Under what conditions can you be sure that the solution set of the system of equations $f(x, y, z) = 0$, $g(x, y, z) = 0$ is a smooth curve without singularities (i.e., a 1-manifold without boundary)?

4. Show that the upper hemisphere of $S^{n-1}(a)$, defined by the equation

$$E_+^{n-1}(a) = S^{n-1}(a) \cap \mathbb{H}^n,$$

is an $n-1$ manifold. What is its boundary?

∗ 5. Let $\mathcal{O}(3)$ denote the set of all orthogonal 3×3 matrices, considered as a subspace of $\mathbb{R}^9$.

(a) Define a $\mathcal{C}^\infty$ function $f : \mathbb{R}^9 \to \mathbb{R}^6$ such that $\mathcal{O}(3)$ is the solution set of the equation $f(\mathbf{x}) = \mathbf{0}$.

(b) Show that $\mathcal{O}(3)$ is a compact 3-manifold in $\mathbb{R}^9$ without boundary.

Hint: Show the rows of $Df(\mathbf{x})$ are independent if $\mathbf{x} \in \mathcal{O}(3)$.

6. Let $\mathcal{O}(n)$ denote the set of all orthogonal $n \times n$ matrices, considered as a subspace of $\mathbb{R}^N$, where $N = n^2$. Show $\mathcal{O}(n)$ is a compact manifold without boundary. What is its dimension?