Lons 1.6, 2.1 Hw 5 on Web/Print

$L\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum \alpha_i L(x_i)$

claim $L\left(\sum_{i=1}^{n} \alpha_i x_i\right) \underset{(2)}{=} \sum \alpha_i L(x_i)$

Pf $L(\sum \alpha_i x_i) \overset{(1)}{=} \sum L(\alpha_i x_i)$

Check 3 things

prop Given function $L: V \to W$

$(L$ is a LnTr$) \iff \forall x, y, c,\ L(cx+y) = cL(x) + L(y)$

positive: check 1 property Pf $\Rightarrow$ ✓

$\Leftarrow$ Suppose $L(cx+y) = cL(x) + L(y)$

~~prop~~ 1 If $c=1 \Rightarrow$ ~~prop~~ #1.

If $y=0, x=0, c=1 \Rightarrow \cancel{L(0)} = L(0) + \cancel{L(0)}$

$0 = L(0)$

~~prop~~ 2 $\Rightarrow$ If $y=0$: $L(cx) = cL(x)$ □

~~example~~ 1. $L: \mathbb{R}^2 \to \mathbb{R}^3$ by $L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a-b \\ 0 \\ a+7b \end{pmatrix}$

□

2. $L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 \\ |b| \\ a+1 \end{pmatrix}$ not linear: $L(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$L(cx+y) = L\begin{pmatrix} ca+a' \\ cb+b' \end{pmatrix} = \begin{pmatrix} (ca+a')^2 \\ |cb+b'| \\ ca+a' \end{pmatrix}$.

$\overset{?}{=} c\begin{pmatrix} a^2 \\ |b| \\ a \end{pmatrix} + \begin{pmatrix} a'^2 \\ |b'| \\ a' \end{pmatrix} = cL(x) + L(y)$

$(ca+a')^2 = ca^2 + a'^2$ ✗

$|cb+b'| = c|b| + |b'|$ ✗

$ca+a' = ca+a'$ ✓

3. $D: P_n \to P_{n-1}$ | $\hat{x}: P_n \to P_{n+1}$ * in quantum mechanics

$Df := \frac{df}{dx}$ | multiplication by x

$D(x^3 - 7x) = 3x^2 - 7$ | $\hat{x} \cdot f = x \cdot f$

momentum | $\hat{x}(7x) = 7x^2$ position

3a is linear $D(f+g) \overset{?}{=} D(f) + D(g)$

$\overset{\shortparallel}{(f+g)'}$ $\overset{?}{=}$ $f' + g'$

(Uniqueness) 10/22/14

$\Rightarrow$ Every $f \in P_n(F)$ can be written in a unique way as a lin combination of the P_i's.

Properties of Basis

<u>Claim</u> suppose $g \in P_n(F)$ satisfies $g(x_i) = y_i$ then $g = P$

<u>Proof</u> By prior claim, $\exists\ d_i \in F$ satisfies

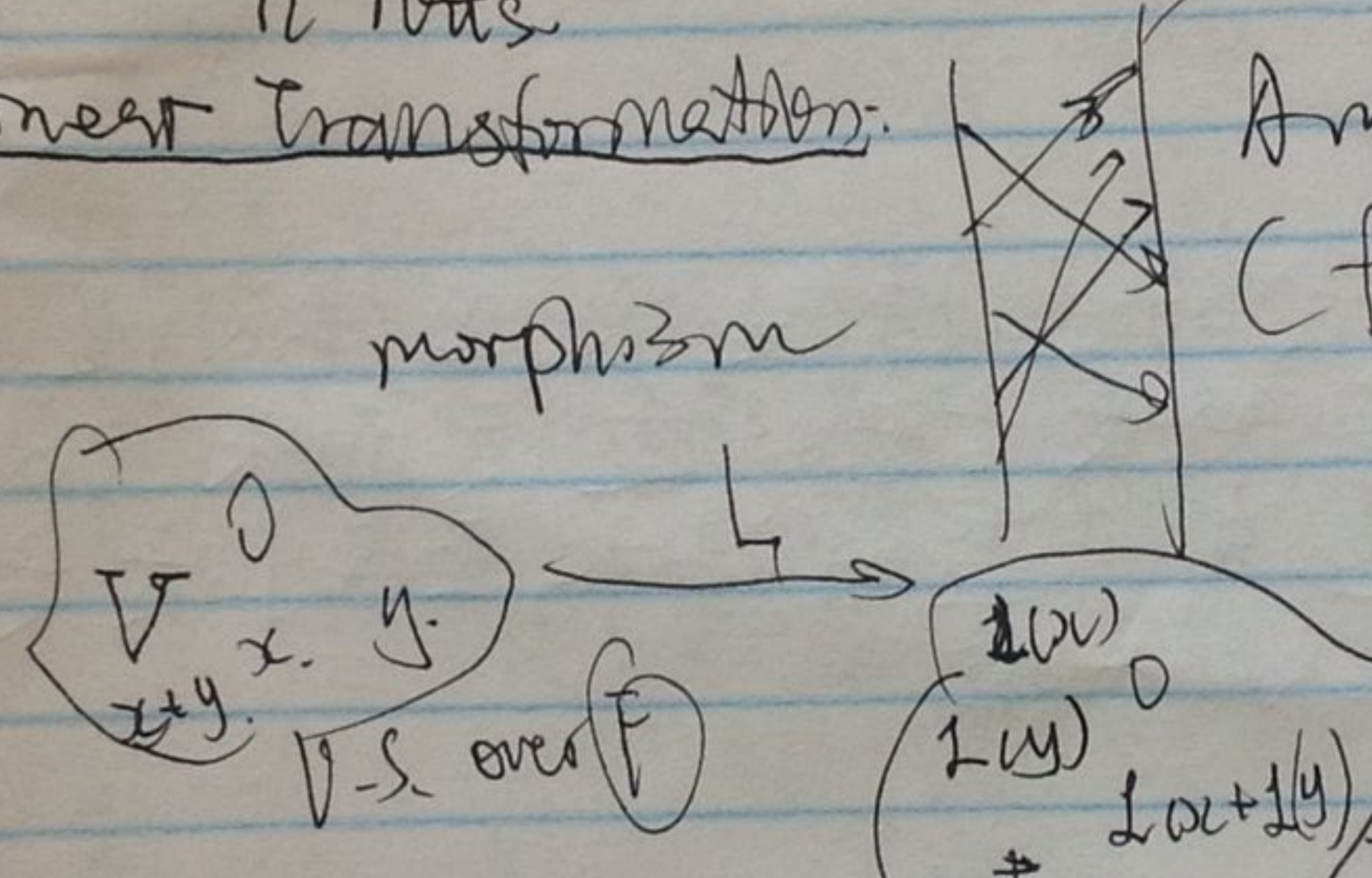
$g = \sum d_i P_i$ but $y_k = g(x_k) = d_k$

So $\forall k\ y_k = d_k$ so $g = \sum d_i P_i = \sum y_i P_i = P$

<u>Aside</u> If $\forall P(x_i) = 0$ then P itself is 0

<u>I.e.</u> If P is n degree polynomial not equal to 0, then P has at most n roots.

<u>Linear Transformation</u>: A map between 2 objects of the (f) kind that respect their structures.



0. $L(0_V) = 0_W$

1. $\forall x, y \in V$, $x +_V y$
$L(x +_V y) = L(x) +_W L(y)$

2. $\forall x \in V, \forall c \in F$
$L(cx) = c \cdot L(x)$

<u>Def</u> Let V & W be V.S. over some field F. A Linear Transformation from V to W is a map (function) $L: V \to W$ satisfies

1. $\forall x, y \in V$, $L(x+y) = L(x) + L(y)$

2. $\forall x \in V$, $c \in F$, $L(cx) = c \cdot L(x)$

<u>Claim</u> If L is a ~~Lin~~ then $L(0) = 0$

~~Pf~~ $L(0_V) = L(0_F \cdot 0_V) \overset{2}{=} 0_F L(0_V) = 0_W$ $\square$

<u>Claim</u> $L(x-y) = L(x) - L(y)$

~~Pf~~ $L(x-y) = L(x + (-1)y) \underset{(1)}{=}$

$L(x) + L((-1)y) \underset{(2)}{=} L(x) + (-1)L(y) = L(x) - L(y)$

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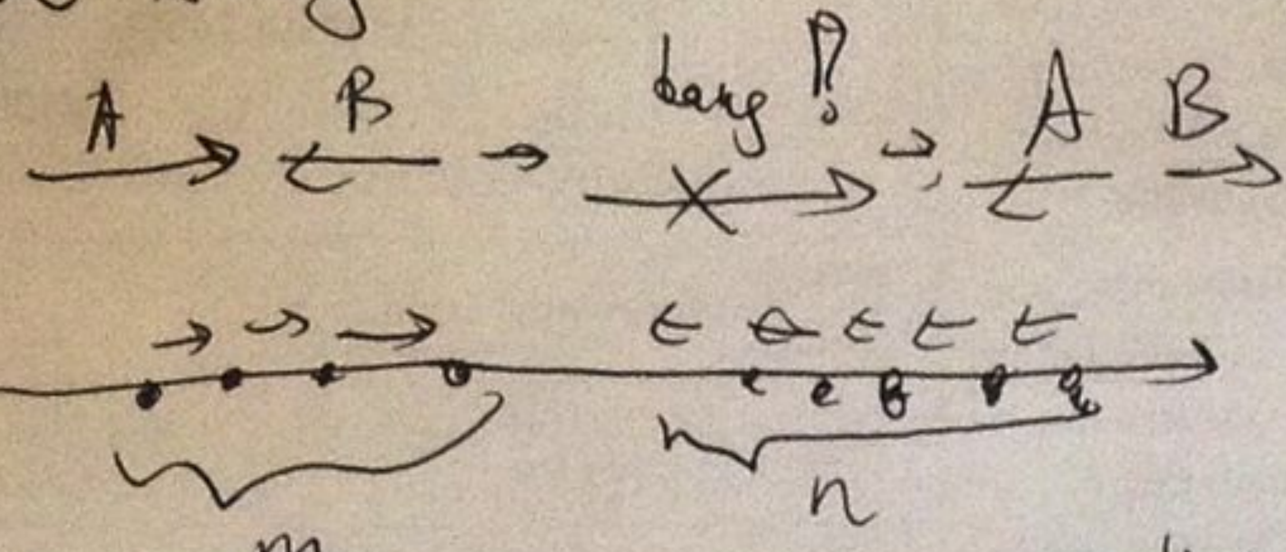
Reading Along 1.6, 2.1 Hw 5 on Web/Print

Riddle Along: counting collisions.



To try: Lagrange Interpolation Linear Algebra

Reminder: $x_1 \dots x_{n+1} \in F$ distinct

$y_1 \dots y_{n+1} \in F$ any

$\exists ! \, P \in P_n(F)$ s.t. $P(x_i) = y_i$?

$P_i(x) = \prod_{j \neq i} (x - x_j)$

$P_i(x_k) = \begin{cases} 0, & k \neq i \\ \neq 0, & k = i \end{cases}$ ~~Now set~~

$P = 5P_1 + 2P_2 + 2P_3 = x^2 - 4x + 5$

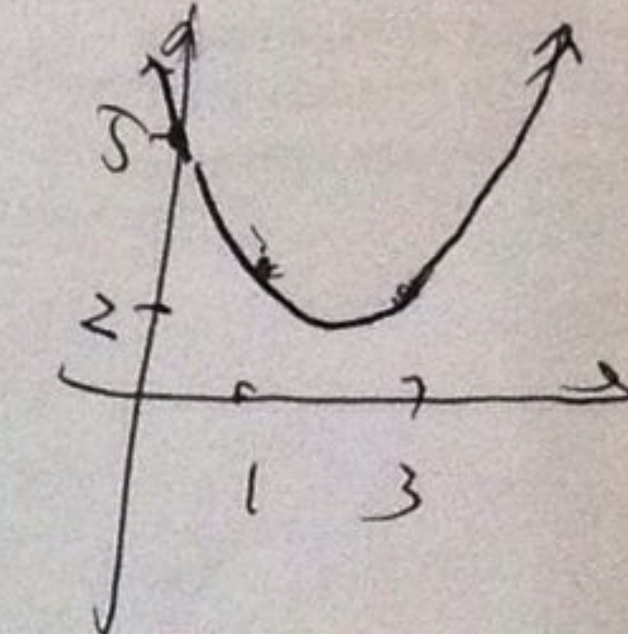
~~Example~~ $x_{1,2,3} = 0, 1, 3$ $y_{1,2,3} = 5, 2, 2$

$P(0) = 5$ $P(1) = 2 = P(3)$

$P_1 = (x - x_2)(x - x_3) = (x-1)(x-3) = x^2 - 4x + 3$

$P_2 = (x - x_1)(x - x_3) = x^2 - 3x$

$P_3 = (x - x_1)(x - x_2) = x^2 - x$



set $\tilde{P}_i(x) = \dfrac{P_i(x)}{P_i(x_i)} = \dfrac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$

$\tilde{P}_i(x_k) = \begin{cases} 0, & k \neq i \\ 1, & k = i \end{cases}$

Now set $P(x) = \sum_{i=1}^{n+1} y_i \tilde{P}_i(x)$

Indeed

$P(x_k) = \sum y_i \tilde{P}_i(x_k)$

$= 0 + 0 + 0 + \dots y_k \tilde{P}_k(x_k)$

$= y_k \cdot 1 = y_k$

Uniqueness: $\beta = \{\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \dots, \tilde{P}_{n+1}\}$ is lin independent.

Indeed, suppose $\sum \alpha_i \tilde{P}_i = 0$

compute at x_k: $0 + 0 \dots + 0 + \alpha_k \tilde{P}_k(x_k) = 0$

$\Rightarrow \alpha_k = 0 \Rightarrow \forall_k \; \alpha_k = 0$

$\dim P_n(F) = n+1 \Rightarrow \beta$ is a basis

$(f+g)' \stackrel{?}{=} f'+g'$ Prop 1 holds

true by basic calculus

Prop 2 $(cf)' = c \cdot f'$

$A \in M_{m\times n}(F)$ $\quad A=(a_{ij}) = \begin{pmatrix} a_1 & \cdots & a_{1n}\\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$

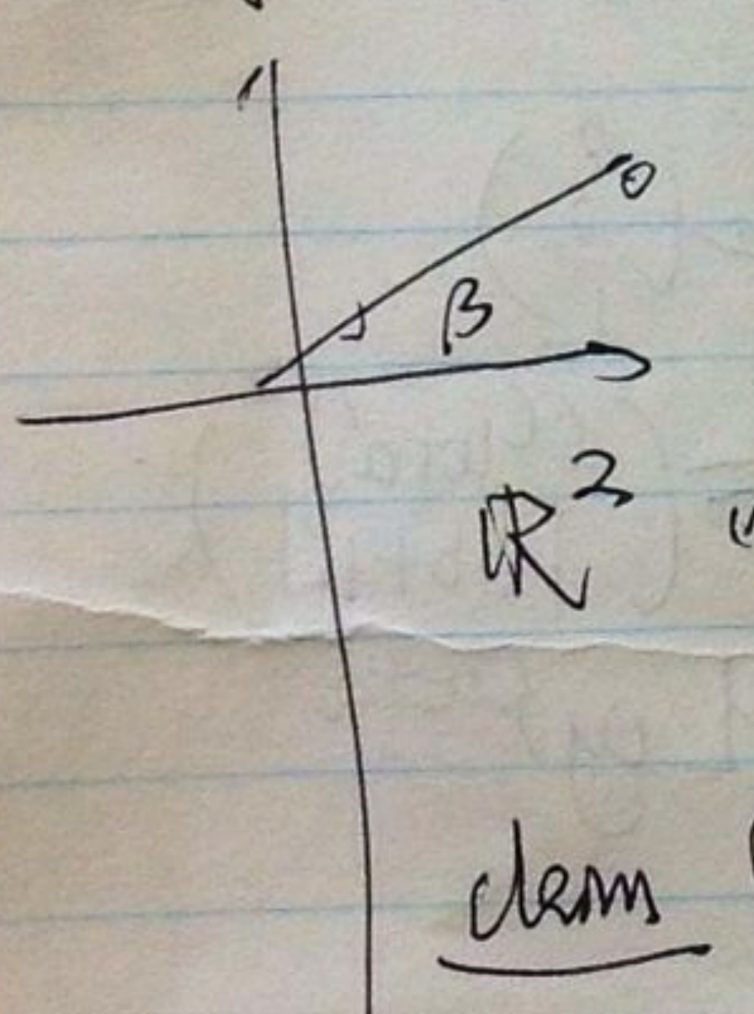
Define $L_A : F^n \to F^m$ by $x = \begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix} \mapsto \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots a_{mn}x_n\end{pmatrix}$

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m\end{pmatrix}$ $\quad y_j = \sum_{i=1}^n a_{ji} x_i$

LHS $L(cx+z)_j = \sum_{i=1}^n a_{ji}(cx+z)_i = \sum_{i=1}^n a_{ji}(cx_i+z_i)$

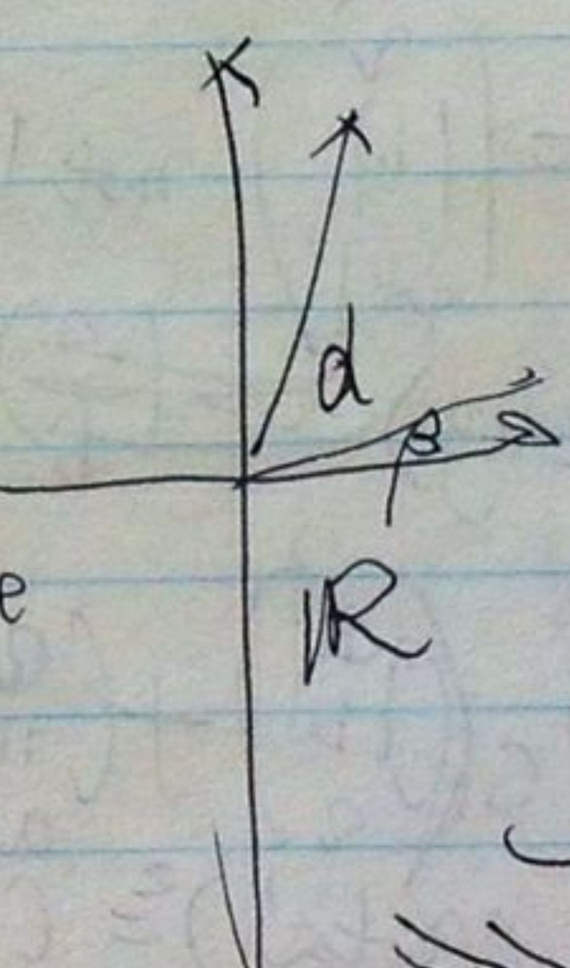
RHS: $[cL(x)+L(z)]_j = c\sum_{i=1}^n a_{ji}x_i + \sum_{i=1}^n a_{ji}z_i$

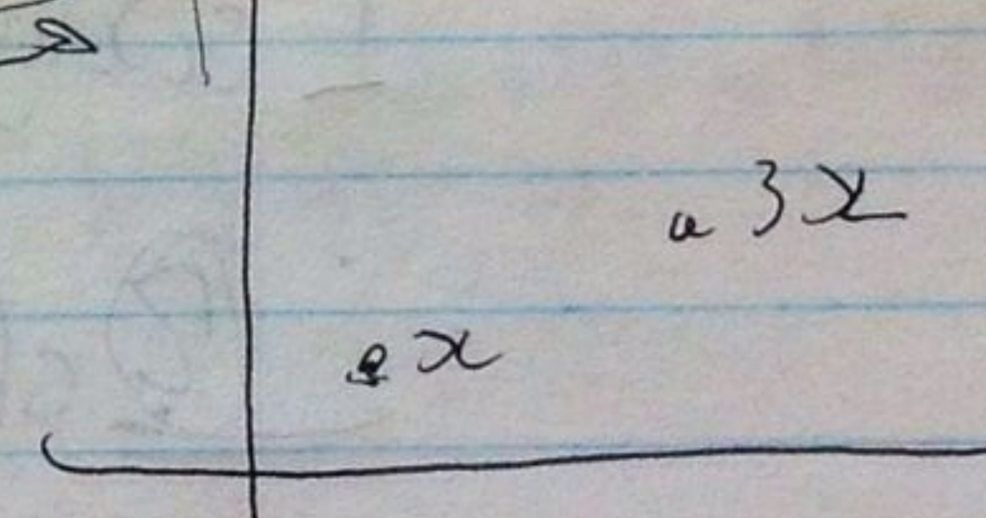
If LHS $\stackrel{?}{=}$ RHS ! Distributive Law



$\xrightarrow{R_\alpha}$

"rotating by α counter clockwise"



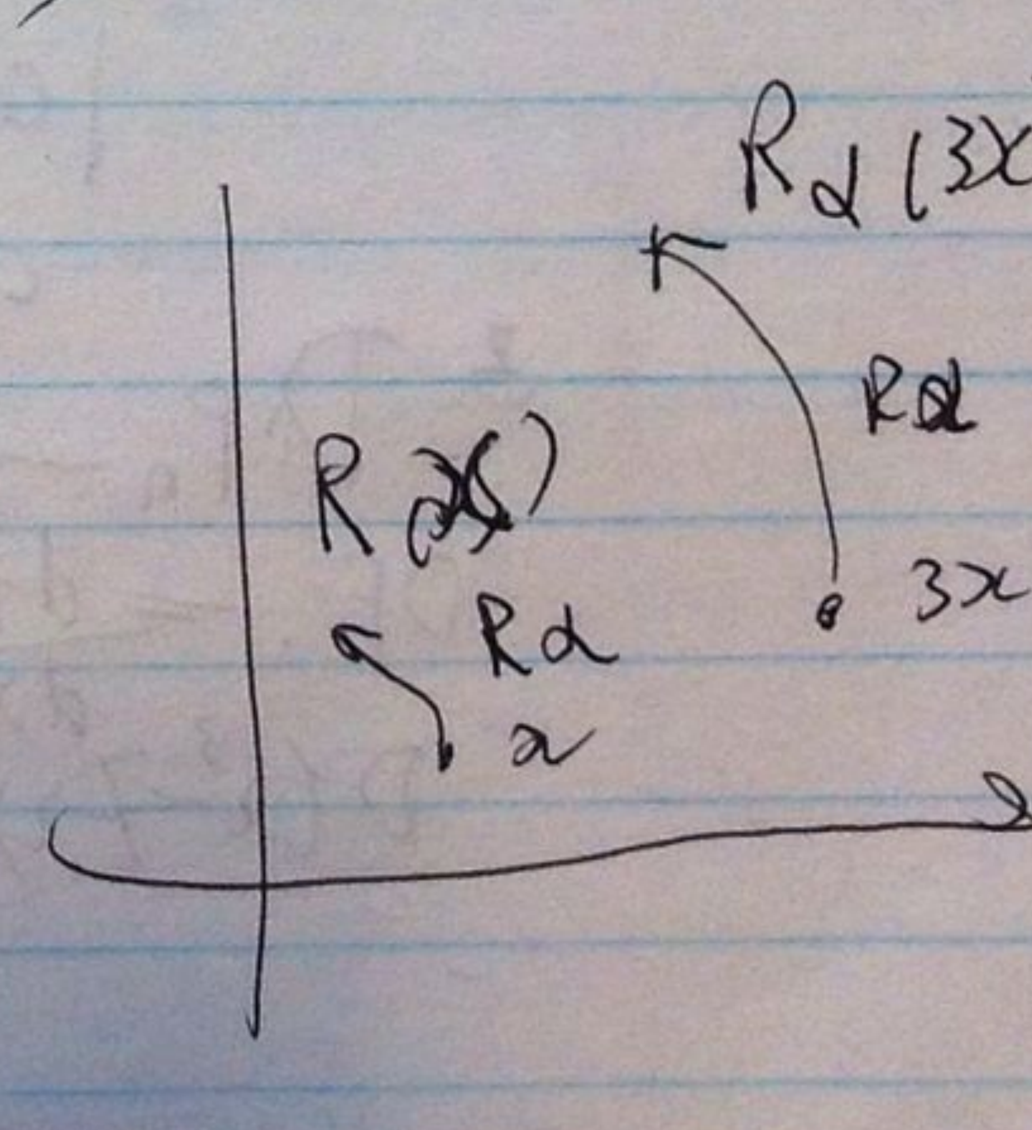


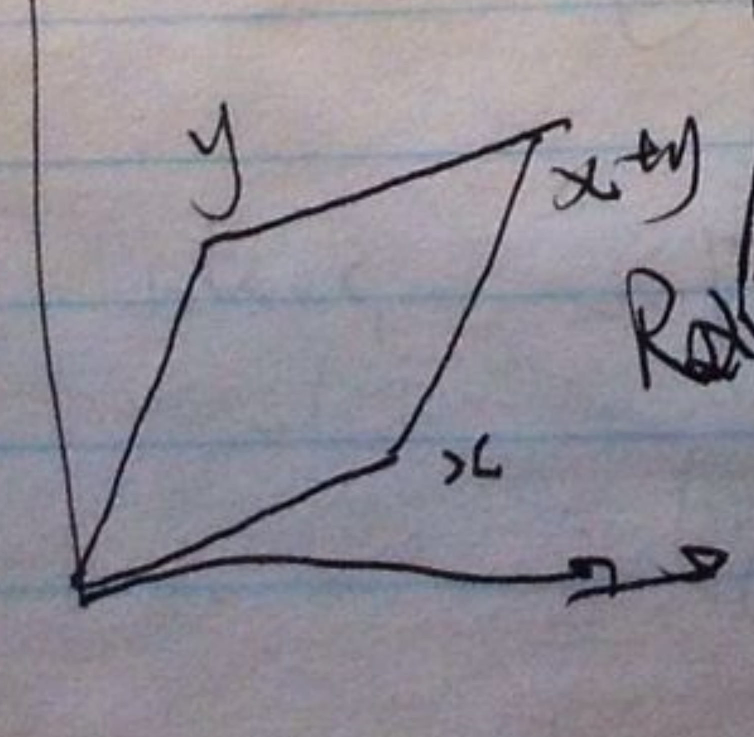
claim R_α is linear

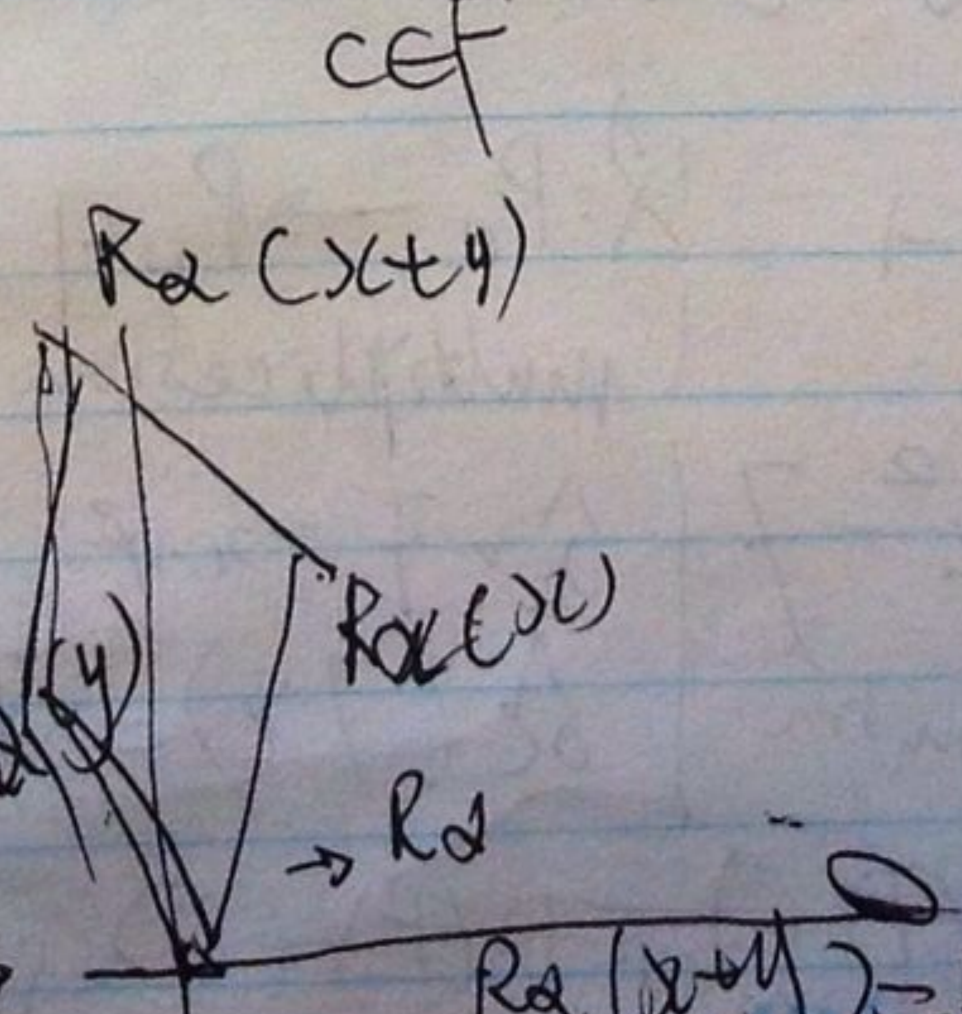
$\Rightarrow$

Pf in pictures $\quad$ 2. $R_\alpha(3x) = 3R_\alpha(x)$

$c \in F$







$R_\alpha(x+y) = R_\alpha(x) + R_\alpha(y)$