

Problem Set 11 — MAT257

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Disclaimer—This page has been typeset by a student as a *convenient consolidation* of the homework problems. There inevitably will be mistakes; always defer to the official handout!

* **Update 23/01/2017:** Problem 1 in §20 was initially typed up as problem 2. It has been corrected now.

Problems marked with * are to be submitted for credit.

1 Munkres §17 (p.151)

- * 6. Let S be the tetrahedron in \mathbb{R}^3 having vertices $(0, 0, 0)$, $(1, 2, 3)$, $(0, 1, 2)$, and $(-1, 1, 1)$. Let $f : S \rightarrow \mathbb{R}$ be given by

$$f(x, y, z) = x + 2y - z.$$

Evaluate $\int_S f$.

Hint: Use a suitable linear transformation g as a change of variables.

7. Let $0 < a < b$. If one takes the circle in the xz -plane of radius a centred at the point $(b, 0, 0)$, and if one rotates it about the z -axis, one obtains a surface called the **torus**. If one rotates the corresponding circular disc instead of the circle, one obtains a 3-dimensional solid called the **solid torus**. Find the volume of this solid torus.

Hint: One can proceed directly, but it is easier to use the **cylindrical coordinate transformation**

$$g(r, \theta, z) = (r \cos \theta, r \sin \theta, z).$$

The solid torus is the image under g of the set of all (r, θ, z) for which

$$\begin{aligned}(r - b)^2 + z^2 &\leq a^2, \\ 0 &\leq \theta \leq 2\pi.\end{aligned}$$

2 Munkres §20 (p.167)

1. Show that if h is an orthogonal transformation, then h carries every orthonormal set to an orthonormal set.
- * 2. Find a linear transformation $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserves volumes but is not an isometry.

3 Munkres §21 (p.187)

- * 3. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the function $h(\mathbf{x}) = \lambda\mathbf{x}$. If \mathcal{P} is a k -dimensional parallelepiped in \mathbb{R}^n , find the volume of $h(\mathcal{P})$ in terms of the volume of \mathcal{P} .
5. Prove the following:
Theorem. Let W be an n -dimensional vector space with an inner product. Then there exists a unique real-valued function $V(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of k -tuples of vectors of W such that:
- (i) Exchanging \mathbf{x}_i with \mathbf{x}_j does not change the value of V .
 - (ii) Replacing \mathbf{x}_i by $\mathbf{x}_i + c\mathbf{x}_j$ (for $j \neq i$) does not change the value of V .
 - (iii) Replacing \mathbf{x}_i by $\lambda\mathbf{x}_i$ multiplies the value of V by $|\lambda|$.
 - (iv) If the \mathbf{x}_i are orthonormal, then $V(\mathbf{x}_1, \dots, \mathbf{x}_k) = 1$.

Proof:

- (a) Prove uniqueness. [*Hint:* Use the Gram-Schmidt process.]
- (b) Prove existence. [*Hint:* If $f : W \rightarrow \mathbb{R}^n$ is a linear transformation that carries an orthonormal basis to an orthonormal basis, then f carries the inner product on W to the dot product on \mathbb{R}^n .]

4 Munkres §22 (p.193)

1. Let A be open in \mathbb{R}^k ; let $\alpha : A \rightarrow \mathbb{R}^n$ be of class C^r ; let $Y = \alpha(A)$. Suppose $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry; let $Z = h(Y)$ and let $\beta = h \circ \alpha$. Show that Y_α and Z_β have the same volume.
- * 2. Let A be open in \mathbb{R}^k ; let $f : A \rightarrow \mathbb{R}$ be of class C^r ; let Y be the graph of f in \mathbb{R}^{k+1} , parametrized by the function $\alpha : A \rightarrow \mathbb{R}^{k+1}$ given by $\alpha(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$. Express $v(Y_\alpha)$ as an integral.

Hint: In some approaches, it may be possible to simplify the end result using the formula

$$\det(I_{n \times n} + vw^T) = 1 + v^T w$$

which holds for vectors $v, w \in \mathbb{R}^n$. — **DBN**

5 “In addition...”

A. Curves!

Consider a simple curve given by a smooth map $\gamma : [0, T] \rightarrow \mathbb{R}^n$, and let $\mathcal{C} := \text{Im}(\gamma)$.

1. Prove that the arclength $\lambda(\mathcal{C})$ of the curve \mathcal{C} is given by

$$\lambda(\mathcal{C}) = \int_0^T \|\dot{\gamma}(t)\| dt$$

where $\|\cdot\|$ is the Euclidian norm on \mathbb{R}^n and $\dot{\gamma} = \frac{d}{dt}\gamma$.

2. Consider the length at time $t \in [0, T]$ of γ ,

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du.$$

Let $\hat{\gamma} : [0, \lambda(\mathcal{C})] \rightarrow \mathbb{R}^n$ be the reparametrization of \mathcal{C} in terms of arclength (i.e. $\mathcal{C} = \text{Im}(\hat{\gamma})$). Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous in a neighbourhood of \mathcal{C} ; express the integral f in terms of the parametrizations γ and $\hat{\gamma}$.

3. Consider the helix $\mathcal{C} \subset \mathbb{R}^3$ parametrized as follows:

$$\gamma(t) = \begin{pmatrix} r \cos t \\ r \sin t \\ 2t \end{pmatrix}, \quad t \in [-2\pi, 2\pi].$$

Compute the arclength $\lambda(\mathcal{C})$ and the integral $\int_{\mathcal{C}} f$ where $f(x, y, z) = xy \sin z$.

B. Surfaces!

A parametrization of an orientable surface $\mathcal{S} \subset \mathbb{R}^3$ is a \mathcal{C}^r map

$$\begin{aligned} \sigma : \mathcal{U} &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto (x(u, v), y(u, v), z(u, v)) \end{aligned}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is open.

* 1. Compute the area $\mathcal{A}(\mathcal{S})$ of the surface $\mathcal{S} \subset \mathbb{R}^3$ given by the parametrization

$$\sigma(u, v) = (u \cos v, u \sin v, v)$$

$$\forall (u, v) \in (0, 1) \times (0, \pi).$$

2. Let \mathcal{S} be the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = b^2$ where $0 < b < a$. Compute the surface area $\mathcal{A}(\mathcal{S})$.