MAT240 WEEK LEC Oct. 30, 2012

Fix a LT $T: V \to W$

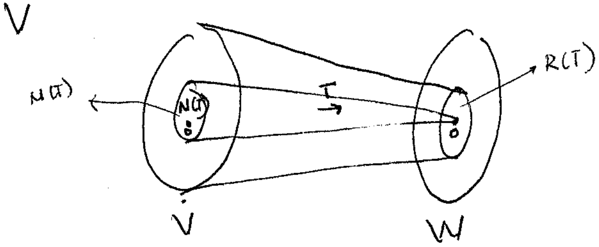
Definitions:

$N(T) = \ker T = \{ v \in V : Tv = 0 \} \subset V$

"null space" "kernel"

"range" "image"

$R(T) = \operatorname{im} T = \{ Tv : v \in V \} \subset W$




PROPOSITION / DEFINITION

1. $N(T)$ is a subspace of $V$ nullity $(T) = \dim (N(T))$
2. $R(T)$ is a subspace of $W$ rank $(T) = \dim R(T)$

Examples:

1. $T = 0_{L(V,W)}$ linear transformation of V to W that maps everything to 0.

$Tv = 0$

$\ker T = N(T) = V$ nullity $(T) = \dim V$

$\operatorname{im} T = R(T) = \{0\}$ rank $(T) = \dim(\operatorname{im}(T)) = 0$.

2. $V = W$; $T = I$ $Tv = v$.

$\ker(T) = N(T) = \{0\}$ nullity $(T) = 0$. (the only thing that maps to 0 is 0 itself)

$\operatorname{im}(T) = R(T) = V$ rank $(T) = \dim V$ (nullity ↑ ⇒ rank ↓)

3. $V = P_n(\mathbb{R}) = W$ $T = \frac{d}{dx}$ $Tx^3 = 3x^2$

* the only thing whose deriv is 0 are the constants, so:

$\ker(T) = N(T) = \{ c \cdot x^0 : c \in \mathbb{R} \}$ nullity $= 1$

$\operatorname{im}(T) = R(T) = P_{n-1}(\mathbb{R})$ rank $(T) = n$.

sum $= n + 1 = \dim V$

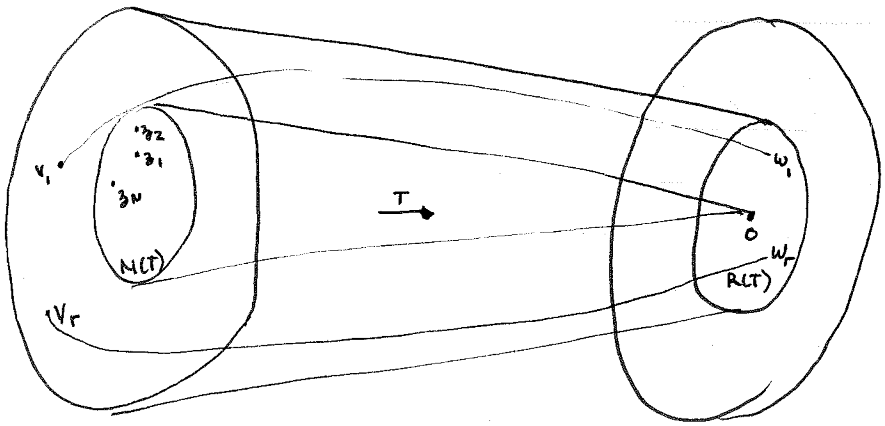
(sum of N and R → complementary)

**Theorem** "DIMENSION / RANK - NULLITY THM"

Given $T: V \to W$, $\dim V = \text{rank}(T) + \text{nullity}(T)$

null space: collection of all things that map to 0.

→ The more T contracts things to 0, the less the img will be.




$V_1 \to w_1$
$V_r \to w_r$

PROOF OF DIM THM.

Choose some basis $z_1 \cdots z_n$ of $N(T)$. [This is possible because $N(T)$ is a subspace of a finite-dimensional space].
$\{z_i\}$ is LI in $V$, so it can be extended by adding $v_1 \cdots v_r$ to become a basis of $V$.

Let $w_i = T(v_i) \in R(T)$. (claim: $\{w_i\}$ are a basis of $R(T)$)

CLAIM 1 $\{w_i\}$ is LI.

PROOF Assume $\sum_{i=1}^{r} b_i w_i = 0$ in $W$.
then $0 = \sum b_i w_i = \sum b_i T(v_i) = T(\sum b_i v_i)$
$\Rightarrow \sum b_i v_i \in N(T)$ (kernel $\approx$ null space)
So for some constant $a_i \in F$, $\sum b_i v_i = \sum a_j z_j$
$\Rightarrow \sum b_i v_i - \sum a_j z_j = 0$, but $\{z_j\} \cup \{v_i\}$ is LI,
so $\forall i\ b_i = 0$ & $\forall j\ a_j = 0$. So $\{w_i\}$ are LI. ✓

CLAIM 2 $\{w_i\}$ span $R(T)$

PROOF Let $w \in R(T)$, meaning find $v \in V$, st. $w = T(v)$
as $\{z_j\} \cup \{v_i\}$ is a basis, we can find $a_j$'s & $b_i$'s st.
$v = \sum a_j z_j + \sum b_i v_i$, so
$w = T(v) = T(\text{-}\text{''}\text{-}) = \sum a_j T(z_j) + \sum b_i T(v_i)$
$= 0 + \sum b_i w_i$ ✓
$\hookrightarrow$ $w$ is a LC of the $\{w_i\}$'s.

$\dim V = |\{z_i\} \cup \{v_i\}| = n + r$

$\text{rank}(T) = \dim R(T) = |\{w_i\}| = |\{v_i\}| = r.$

$\text{nullity}(T) = \dim N(T) = |\{z_i\}| = n.$ ■

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SUMMARY

$T: V \to W$, $V$ f.d., nullity $= \dim(\ker)$

rank $= \dim(\text{im})$

$\dim V = \text{rank}(T) + \text{nullity}(T)$

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COROLLARY $T$ is 1-1 $\Leftrightarrow$ $\ker(T) = \{0\}$ $\Leftrightarrow$ nullity$(T) = 0$

PROOF ($\Rightarrow$) if $T$ is 1-1 and $Tv = 0 = T0$ so $Tv = T0$ $\Rightarrow$ so $v = 0$.

$\therefore \ker(T) = 0.$

($\Leftarrow$) suppose $\ker(T) = 0$, and $Tv_1 = Tv_2$. $\Rightarrow Tv_1 - Tv_2 = T(v_1 - v_2)$

$\Rightarrow v_1 - v_2 \in \ker(T)$

$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$ ✓

* if nullity $= 0$, kernel is empty, so 1-1

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COROLLARY rank $T = \dim W$ $\Leftrightarrow$ im $T$ $\leftrightarrow$ $T$ is onto.

subspace has max dim, iff it is the whole thing.
image is "everything" = transformation is onto. } by def$^n$. no proof needed

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• COROLLARY of Thm 1

If $\dim V = \dim W$, then TFAE (the ff are equivalent)

1) $T$ is 1-1
2) $T$ is onto
3) rank $T = \dim V$
4) $T$ is invertible

$$\begin{tikzcd} V \arrow[r, shift left, "T"] & W \arrow[l, shift left, "S=T^{-1}"] \end{tikzcd}$$

2) $\Leftrightarrow$ 3) ✓, 1$\Leftrightarrow$3 | $T$ is 1-1 $\Leftrightarrow$ null $= 0$ $\Leftrightarrow$ as $n + r = \dim V$, $\overset{\text{iff}}{\Leftrightarrow}$ $\overbrace{\text{rank } T = \dim V}^{T \text{ is onto.}}$

invertible $\overset{\Rightarrow}{\text{implies}}$ 1-1 & onto. so 4) $\Rightarrow$ 1) and 2).

1-1 $\Rightarrow$ onto $\Rightarrow$ invertible

onto $\Rightarrow$ 1-1 $\Rightarrow$ invertible ■

depends on context.

Theorem 2 $T: V \to W$ & $T': V' \to W'$ are "isomorphic" iff $(\dim V, \dim W, \text{rank } T) = (\dim V', \dim W', \text{rank } T')$

ISOMORPHIC:

$$\begin{tikzcd} V \arrow[r, bend left, "\Phi"] & V' \arrow[l, bend left, "\Phi^{-1}"] \end{tikzcd}$$

$\Phi$ preserves structures $V$ has.

$\Phi$ is an isomorphism if it's a LT, & 1-1 & onto.

So for:

"commutes"

(range →)

$$\begin{tikzcd} u \quad V \arrow[r, "\Phi_1"] \arrow[d, "T"'] & V' \quad \Phi_1 u \arrow[d, "T'"] \\ Tu \quad W \arrow[r, "\Phi_2"] \arrow[r, bend right, "\Phi_2 Tu"'] & W' \quad \Phi_2 u \end{tikzcd}$$

- $\Phi_2, \Phi_1$ is a LT (preserve addition & mult of a scalar in $W$, & $V$, respectively)

- $\left\{ \begin{array}{l} \text{If have a } u \text{ in } V, \text{ img in } W \text{ will be } Tu. \\ \forall u \in V \quad T'\Phi u = \Phi_2 Tu. \end{array} \right.$
- $\Phi_1$ & $\Phi_2$ are invertible.

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## LINEAR TRANSFORMATIONS & MATRICES

Reminder: choosing a basis, $V$ is isomorphic to $F^n$.

Goal: Choosing basis for both $V$ & $W$, $\mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(F)$ $\quad m = \dim W$

$n = \dim V$

Let $\beta = (u_1, \ldots u_n)$ be an ordered basis of $V$. ($V$ is fd)

given $x \in V$, find scaler $a_1 \ldots a_n$ st. $x = \sum a_i u_i$

define $[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ "the coordinates of $x$ rel to $\beta$"

EXAMPLE in $P_2(\mathbb{R})$

$[x^2 - 2x + 3]$ $(x^0, x^1, x^2) = \begin{pmatrix} \\ \end{pmatrix}$

$$x^2 - 2x + 3 = \underbrace{3}_{a_1} x^0 + \underbrace{(-2)}_{a_2} x^1 + \underbrace{(1)}_{a_3} x^2$$

(to be cont'd next class)