

Problem Set 9 — MAT257

November 30, 2016

Problems marked with * are to be submitted for credit.

1 Munkres §13 (p.111)

- Let $f, g : S \rightarrow \mathbb{R}$. Assume f and g are integrable over S .
 - Show that if f and g agree except on a set of measure zero, then $\int_S f = \int_S g$.
 - Show that if $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$ and $\int_S f = \int_S g$, then f and g agree except on a set of measure zero.
- Let A be a rectangle in \mathbb{R}^k ; let B be a rectangle in \mathbb{R}^n ; let $Q = A \times B$. Let $f : Q \rightarrow \mathbb{R}$ be a bounded function. Show that if $\int_Q f$ exists, then

$$\int_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y})$$

exists for $\mathbf{x} \in A \setminus D$, where D is a set of measure zero in \mathbb{R}^k .

- Complete the proof of **Corollary 13.4**.
- Let S_1 and S_2 be bounded sets in \mathbb{R}^n ; let $f : S_1 \cap S_2 \rightarrow \mathbb{R}$ be a bounded function. Show that if f is integrable over S_1 and S_2 , then f is integrable over $S_1 \setminus S_2$, and

$$\int_{S_1 \setminus S_2} f = \int_{S_1} f - \int_{S_1 \cap S_2} f.$$

- Let S be a bounded set in \mathbb{R}^n ; let $f : S \rightarrow \mathbb{R}$ be a bounded continuous function; let $A = \text{Int}S$. Give an example where $\int_A f$ exists and $\int_S f$ does not.
- Show that **Theorem 13.6** holds without the hypothesis that f is continuous on S .
- Prove the following:

Theorem. Let S be a bounded set in \mathbb{R}^n ; let $f : S \rightarrow \mathbb{R}$ be a bounded function. Let D be the set of points of S at which f fails to be continuous. Let E be the set of points of $\text{Bd}S$ at which the condition

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = 0$$

fails to hold. Then $\int_S f$ exists if and only if D and E have measure zero.

Proof.

- Show that f_S is continuous at each point $\mathbf{x}_0 \notin D \cup E$.
- Let B be the set of isolated points of S ; then $B \subset E$ because the limit cannot be defined if \mathbf{x}_0 is not a limit point of S . Show that if f_S is continuous at \mathbf{x}_0 , then $\mathbf{x}_0 \notin D \cup (E \setminus B)$.
- Show that B is countable.
- Complete the proof.

2 Munkres §14 (pp.120-121)

1. Let S be a bounded set in \mathbb{R}^n that is the union of the countable collection of rectifiable sets S_1, S_2, \dots .

(a) Show that $S_1 \cup \dots \cup S_n$ is rectifiable.

(b) Give an example showing that S need not be rectifiable.

2. Show that if S_1 and S_2 are rectifiable, so is $S_1 \setminus S_2$, and

$$v(S_1 \setminus S_2) = v(S_1) - v(S_1 \cap S_2).$$

3. Show that if A is a nonempty, rectifiable open set in \mathbb{R}^n , then $v(A) > 0$.

* 4. Give an example of a bounded set of measure zero that is rectifiable, and an example of a bounded set of measure zero that is not rectifiable.

5. Find a bounded closed set in \mathbb{R} that is not rectifiable.

6. Let A be a bounded open set in \mathbb{R}^n ; let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded continuous function. Give an example of where $\int_{\bar{A}} f$ exists but $\int_A f$ does not.

7. Let S be a bounded set in \mathbb{R}^n .

(a) Show that if S is rectifiable, then so is the set \bar{S} , and $v(S) = v(\bar{S})$.

(b) Give an example where \bar{S} and $\text{Int}S$ are rectifiable, but S is not.

8. Let A and B be rectangles in \mathbb{R}^k and \mathbb{R}^n , respectively. Let S be a set contained in $A \times B$. For each $\mathbf{y} \in B$, let

$$S_{\mathbf{y}} = \{\mathbf{x} : \mathbf{x} \in A \text{ and } (\mathbf{x}, \mathbf{y}) \in S\}.$$

We call $S_{\mathbf{y}}$ a **cross-section** of S . Show that if S is rectifiable, and if $S_{\mathbf{y}}$ is rectifiable for each $\mathbf{y} \in B$, then

$$v(S) = \int_{\mathbf{y} \in B} v(S_{\mathbf{y}}).$$

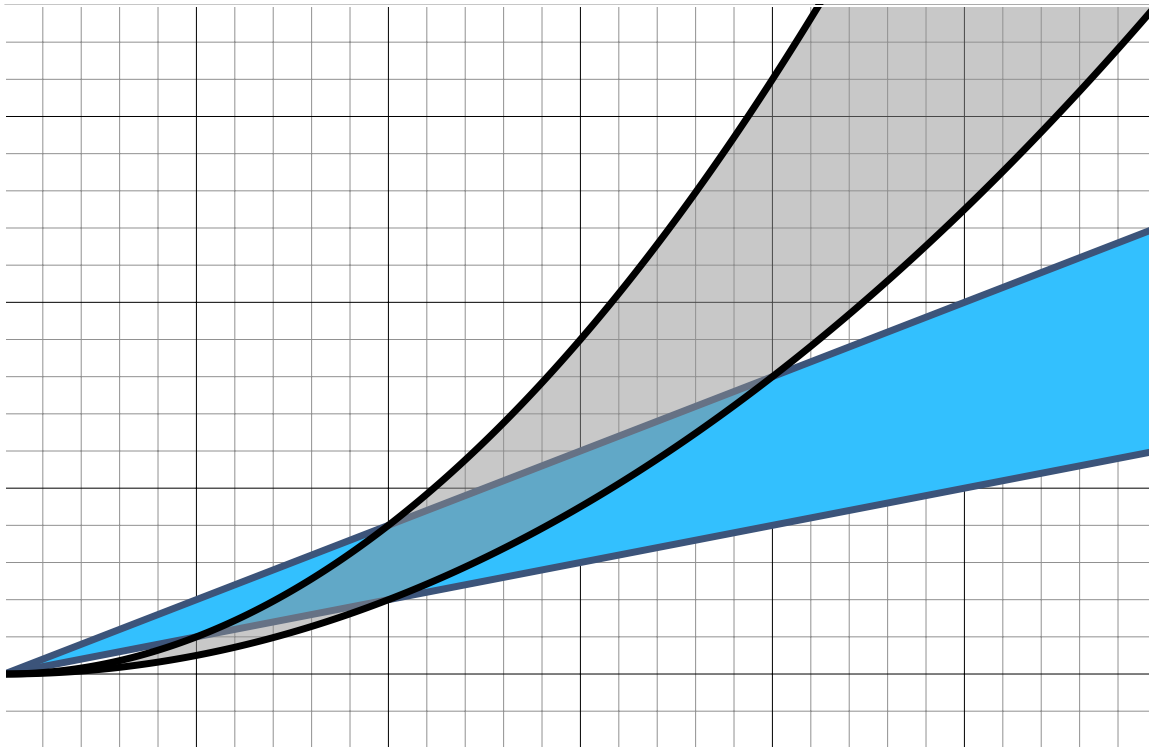
3 Munkres §15 (pp.132-133)

* 4. Let $f(x, y) = 1/(y+1)^2$. Let A and B be the open sets

$$A = \{(x, y) : x > 0 \text{ and } x < y < 2x\},$$

$$B = \{(x, y) : x > 0 \text{ and } x^2 < y < 2x^2\},$$

of \mathbb{R}^2 . Show that $\int_A f$ does not exist. Show that $\int_B f$ does exist and compute it.



- * 8. Let A be open in \mathbb{R}^n . We say $f : A \rightarrow \mathbb{R}$ is **locally bounded** on A if each $\mathbf{x} \in A$ has a neighbourhood on which f is bounded. Let $\mathcal{F}(A)$ be the set of all functions $f : A \rightarrow \mathbb{R}$ that are locally bounded on A and continuous on A except on a set of measure zero.
- Show that if f is continuous on A , then $f \in \mathcal{F}(A)$.
 - Show that if f is in $\mathcal{F}(A)$, then f is bounded on each compact subset of A and the definition of the extended integral $\int_A f$ goes through without change.
 - Show that Theorem 15.3 holds for functions f in $\mathcal{F}(A)$.
 - Show that Theorem 15.4 holds if the word “continuous” in the hypothesis is replaced by “continuous except on a set of measure zero.”

4 “In addition...”

- * A. Compute the volume of the “2D ice cream cone”.

$$C = \{(x, y) : |x| \leq y \leq 1 + \sqrt{1 - x^2}\}.$$

- * B. Compute the volume of the “n-dimensional simplex”.