MAT240 - Tutorial. Nov 9, 2006.

$\boxed{2.2}$ 16 $\ker T \subseteq V$ $\beta = \{v_1, \ldots, v_n\}$ $\gamma = \{w_1, \ldots, w_n\}$

Let $\{v_1, \ldots, v_k\}$ be a basis of $N(T)$ extend to $\{v_1, \ldots, v_n\}$.
Look at $\{T(v_{k+1}), \ldots, T(v_n)\} = \tilde{\gamma}$

Claim: $\tilde{\gamma}$ is lin indep.

$$\sum_{i=k+1}^{n} a_i T(v_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} a_i v_i\right) = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} a_i v_i = 0 \quad \because (v_{k+1}, \ldots, v_n \notin \ker T)$$

$$\Rightarrow a_i = 0 \quad i = k+1, \ldots, n.$$

Claim: $\operatorname{span}(\tilde{\gamma}) = R(T)$

~~$\operatorname{span}\{T(v_{k+1}), \ldots, (T(v_n))\} = \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$~~

(1) $\operatorname{span}(\tilde{\gamma}) \subseteq R(T)$

$$\sum_{i=k+1}^{n} a_i T(v_i) = T\left(\underbrace{\sum_{i=k+1}^{n} a_i v_i}_{\in V}\right)$$

(2) $\dim(\operatorname{span}(\tilde{\gamma})) \overset{?}{=} \dim(R(T))$

$$\dim(\operatorname{span}(\tilde{\gamma})) = n-k = \dim(V) - \dim(N(T)) = \dim(R(T)).$$

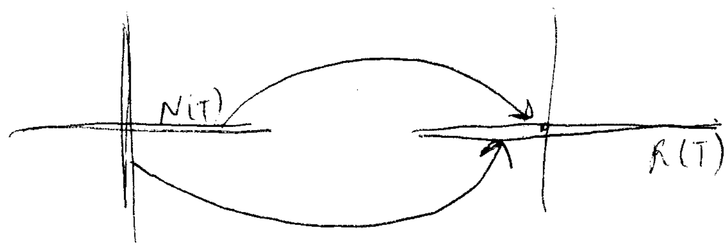
Let $w_i = T(v_i)$ $i = k+1, \ldots, n$
then $\{w_{k+1}, \ldots, w_n\}$ is basis of $R(T)$
extend to $\{w_1, \ldots, w_n\}$.

$$[T]_\beta^\gamma = \left[\begin{array}{ccc|ccc} & & & 0 & & \\ 0 & \cdots & 0 & \vdots & 0 & \\ \hline & & & 1 & & \\ & & & 0 & 1 & 0 \\ & & & \vdots & & \ddots \\ & & & 0 & 0 & 1 \end{array}\right]$$

$k+1$

$\underbrace{\qquad}_{k \text{ cols.}}$

$\boxed{2.1}$ 18.



$T(1,0) = (0,0)$

$T(0,1) = (1,0)$

$T(a,b) = aT(1,0) + bT(0,1) = (b,0)$

$a(0,0)$

$= a, 0$

P.108 16. $\quad C = B^{-1}\,\Phi^{-1}(C)\,B$

$\Phi^{-1}(C) = BCB^{-1}$

Define $\Psi(C) = BCB^{-1}$

Show $\Psi = \Phi^{-1}$.

$\Psi \circ \Phi(A) = \Psi(B^{-1}AB)$

$= B(B^{-1}AB)B^{-1}$

$= A$.

$\Phi \circ \Psi = I$.

17. a)

b). Let $\{v_1, \ldots, v_K\}$ be a basis of V_0.

$\gamma' = \{T(v_1), \ldots, T(v_K)\}$ is a basis of $T(V_0)$

$$\sum_{i=1}^{K} a_i T(v_i) = 0.$$

$$\Rightarrow T\left(\sum_{i=1}^{K} a_i v_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{K} a_i v_i = 0 \quad \forall\ T \text{ is an iso.}$$

$\Rightarrow a_i = 0.$

$\therefore$ lin indep.

& spans: span $\gamma' = T(V_0)$ $\square$.

20. $\beta = \{v_1, \ldots, v_n\}$ $\quad \gamma = \{w_1, \ldots, w_m\}$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi"] & W \arrow[d, "\psi"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd} \qquad \begin{tikzcd} v \arrow[d] & w \arrow[d] \\ {[v]_\beta} & {[w]_\gamma} \end{tikzcd}$$

$\psi \circ T(v) = L_A \circ \phi(v) \quad \forall v$

$\psi \circ T(v)$

$= [T(v)]_\gamma$

$L_A \circ \phi(v)$

$= L_A([v]_\beta)$

$= [T]_\beta^\gamma \cdot [v]_\beta$

$= [T(v)]_\gamma$

$\dim V = n$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi"] & W \arrow[d, "\psi"] \\ F^n \arrow[r, "L_A"] & F^m \end{tikzcd}$$

$\beta = \{v_1, \dots, v_n\}$ $\quad$ $\gamma = \{w_1, \dots, w_m\}$

$v \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\quad$ $v \in V$ $\quad$ $v = \sum_{i=1}^{n} a_i v_i$ $\quad$ $w \mapsto [w]_\gamma$.

Onto: $\phi\left(\sum_{i=1}^{n} a_i v_i\right) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow \phi$ is onto.

1-1: $\phi(v) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$v = \sum_{i=1}^{n} 0 v_i = 0.$

Claim: $\psi \circ T = L_A \circ \phi$

From Q.17 $\quad \psi : V \xrightarrow{iso} W$

$V \supseteq V_0$

$\dim(R(T)) = \dim(\psi(R(T)))$

$\quad \| \qquad\qquad \|$

$r(T) \qquad r(L_A)$

Claim: $\psi(R(T)) = R(L_A)$

Want: (1) $\psi(R(T)) \subseteq R(L_A)$

(2) $\psi(R(T)) \supseteq R(L_A)$

(1) $\psi(T(v))$
$= L_A \circ \phi(v) \in R(L_A)$

(2) $L_A x = L \circ \phi \circ \phi^{-1}(x)$
$= \psi \circ T(\phi^{-1}(x)) \in \psi(R(T))$

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V $\qquad\qquad$ W

$\beta = \{v_1, \dots, v_n\}$ $\qquad$ $\gamma = \{w_1, \dots, w_m\}$.

$$T_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Prove $\Gamma = \{T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis $\mathcal{L}(V, W)$.

$\dim(\mathcal{L}(V, W)) = nm = \#(\Gamma)$.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} T_{ij} = 0$$

Apply to v_K

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} T_{ij}(v_K) = 0$$

$$\sum_{i=1}^{m} a_{iK} T_{iK}(v_K) \quad \because T_{ij}(v_K) = 0,\ K \neq j$$

$$\sum_{i=1}^{m} a_{iK} w_i$$

$\Rightarrow a_{iK} = 0 \quad i = 1, \dots, m.$

$\Rightarrow a_{ij} = 0 \ \forall i,j \ \because K$ is arbitrary

$\therefore$ lin indep

Show $[T_{ij}]_\beta^\gamma = M^{ij}$

$$\| $$

$$\left([T_{ij}(v_1)]_\gamma \middle| \cdots \middle| [T_{ij}(v_n)]_\gamma \right).$$

$$\|$$

$$\begin{array}{c} \left(0 \cdots 0 \,\middle|\, [w_i]_\gamma \,\middle|\, 0 \cdots 0 \right) \\ j^{th} \text{ column} \end{array} = \left(0 \cdots 0 \,\middle|\, e_i \,\middle|\, 0 \cdots 0 \right) = M_{ij}$$

$\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$

$\Phi(T_{ij}) = M^{ij}$

Prove Φ is an iso

We showed $\{T_{ij}\}$ is a basis.

$\{M^{ij}\}$ is a basis of $M_{n\times m}(F)$

$\Rightarrow \Phi$ is an iso.