Homework 2

MAT1100
Due: October 16, 2014 LING-SANG TSE

Solution to Problem 1
a) $n = 11$.

Let $\sigma = (a_{11}a_{12}...a_{1n_1})...(a_{k1}...a_{kn_k})$, where σ is decomposed into disjoint cycles, k is the number of disjoint cycles, and n_i is the length of each disjoint cycle.

The order of the element $\sigma \in S_n$ is the least common multiple of the lengths of the cycles when σ is decomposed into disjoint cycles. In order for the least common multiple to be 18 with minimal $n_1 + ... + n_k$, the n_i's must be pairwise relatively prime. The possible factorizations of 18 into relatively prime factors are (1,18) and (2,9), so the decomposition of σ with minimal $n_1 + ... + n_k$ such that lcm $(n_1, ..., n_k) = 18$ has $n_1 = 2$ and $n_2 = 9$. Then $\sigma = (a_1a_2)(a3..a11) \in S_{11}$. ■

b) Let $\sigma = (a_{11}a_{12}...a_{1n_1})...(a_{k1}...a_{kn_k})$, where σ is decomposed into disjoint cycles, k is the number of disjoint cycles, and n_i is the length of each disjoint cycle. The order of an element $\sigma \in S_n$ is the least common multiple of the lengths of the cycles when σ is decomposed into disjoint cycles, so the maximal order of an element in S_{26} will be given by the least common multiple of all possible partitions of 26. In order for the least common multiple to be maximal, the n_i's must be pairwise relatively prime. One can use a computer algorithm to check that the partition is then $n_1 = 1, n_2 = 4, n_3 = 5, n_4 = 7, n_5 = 9$.

Solution to Problem 2
To show that H is normal in G, we show that $a^{-1}Ha = H$. This is true if and only if given $a \in G$, $aH = Ha$, so we show $aH = Ha$ assuming that H is a subgroup of index 2 in G:

Case 1: If $a \in H$, then $aH = H = Ha$.

Case 2: $a \notin H$.

Since H is a subgroup of index 2 in G, the two left cosets of H are H and aH and the two right cosets of H are H and Ha.

The cosets partition G, so $G = H \cup aH$ and $G = H \cup Ha$, where $\cup$ is a disjoint union. Then aH is precisely all elements of G that are not in H, and Ha is precisely all elements of G that are not in H as well, so $aH = Ha$.

Solution to Problem 3
By the Orbit Stabilizer Theorem, $|C_{S_{20}}(\sigma)| = |S_{20}|/|\operatorname{Orb}(\sigma)|$. From class notes, the orbit of an element σ in S_n is precisely the set of all permutations conjugate to σ, which is precisely the set of all permutations with the same cycle type of σ.

We can use combinatorics to solve the number of elements in S_n with cycle type consisting of one 5-cycle, two 3-cycles, and one 2-cycles:

Using the formula to calculate the number permutations with a specific cycle type in the lecture notes, there are 20!/(5*15!) possible 5-cycles, $15!/(3*3*2*9!)$ possible two 3-cycles with the 15 numbers left over from the 5-cycle, and $9!/2*7!$ possible 2-cycles from the 9 numbers leftover to choose from.

Then the total number of elements in S_n with cycle type consisting of one 5-cycle, two 3-cycles, and one 2-cycles is [20!/(5*15!)]*[$15!/(3*3*2*9!)$]*[9!/2*7!] = 20!/(5*9*2*7!*2).

Then the order of the centralizer of σ is equal to $|S_n|/|S_{20}|/|\operatorname{Orb}(\sigma)|$ = 20!/ [20!/(5*9*2*7!*2)] = 5*9*2*7!*2 = 907200.

Solution to Problem 4

Suppose G is a group of odd order. For any element $x \in G$, $|C_G(x)| = |G|/|\text{Orb }(x)|$, and the conjugacy class of x is the orbit of x under the action of conjugation, so the number of elements in any conjugacy class of G must divide the order of G. Therefore, since G is a group of odd order, then the number of elements in any conjugacy class of G must be odd.

Secondly, there are no elements of order 2, since the order of any element must divide the order of G and G has odd order.

We prove by contradiction. Suppose there exists an element $x \in G$ such that x is conjugate to x^{-1}. $\{x, x^{-1}\}$ cannot be a conjugacy class of G, since any conjugacy class of G must have an odd number of elements. Let y be another element in the conjugacy class of x, so $y = g^{-1}xg$ for some $g \in G$. Then $y^{-1} = g^{-1}x^{-1}g$ is in the conjugacy class of x, since x^{-1} is also in the conjugacy class of x by assumption and y is conjugate to x^{-1}. In other words, for any element y in the conjugacy class of x, y^{-1} is also in the conjugacy class of x, and $y \neq y^{-1}$ since there are no elements of order 2. Then the elements in the conjugacy class of x comes in pairs y and y^{-1}, so there must be an even number of elements in the conjugacy class of x. This is a contradiction, since we established that the the number of elements in any conjugacy class of G must be odd.

Solution to Problem 5

Suppose $G/Z(G)$ is cyclic, so $G/Z(G) = \langle xZ(G) \rangle$ for some $x \in G$. Then for any $yZ(G) \in G/Z(G)$, $yZ(G) = x^nZ(G)$ for some $n \in \mathbb{N}$.

Let $aZ(G), bZ(G) \in G/Z(G)$.

$$\begin{aligned}
ab &= x^n y x^m z \text{ for some } n, m \in \mathbb{N} \text{ and } y, z \in Z(G) \\
&= yzx^n x^m \text{ since y and z are in the centre of G, they commute with all elements in G} \\
&= zyx^m x^n \\
&= x^m z x^n y \\
&= ba
\end{aligned}$$

Therefore, G is abelian.

Solution to Problem 6

If we can show that $G/Z(G)$ is cyclic, then by Problem 5, we can show that G is abelian.

Claim: $G/Z(G)$ is isomorphic to a subgroup of Aut(G).

We define a homomorphism $\phi : G \longrightarrow$ Inn G by $\phi : g \mapsto \phi_g$, where ϕ_g is the inner automorphism conjugation by g. By definition of the map and by definition of Inn G, ϕ is a surjective mapping onto Inn G.

Claim: ker $\phi = Z(G)$.

Proof:

$$\begin{aligned} g \in \text{ ker } \phi &\Leftrightarrow \phi_g \text{ is trivial} \\ &\Leftrightarrow g^{-1}xg = x \text{ for all x in G} \\ &\Leftrightarrow g \in Z(G) \end{aligned}$$

Then by the first isomorphism theorem, $G/\text{ker } \phi \cong \text{im } \phi$, so $G/Z(G) \cong \text{Inn } G \leq \text{Aut } G$ by Problem 4 of Homework 1.

Therefore, $G/Z(G)$ is isomorphic to a subgroup of Aut(G), and since Aut (G) is cyclic and since subgroups of cyclic groups are cyclic, $G/Z(G)$ is also cyclic. Therefore, by Problem 5, G is abelian.

Solution to Problem 7

a) Let $H \leq G$ and let $(G : H) =$ n. Let $g_1H, .., g_nH$ be the cosets of H.

Claim: For any element $g \in G$, and for any $1 \leq i \leq n$, $gg_iH = g_{j_i}H$ for some $1 \leq j_i \leq n$, and if $g_i \neq g_k$, $g_{j_i}H \cap g_{k_i}H = \emptyset$.

Proof: Since the cosets partition G, $gg_i \in g_{j_i}H$ for some $1 \leq j_i \leq n$. Then $g_{j_i}^{-1}gg_iH = H$, so $gg_iH = g_{j_i}H$.

Suppose $g_i \neq g_k$. Then $g_{j_i}H$ and $g_{k_i}H$ are cosets of H. Since $g_jH \cap g_kH = \emptyset$, $gg_j \notin gg_kH = g_{k_i}$, so since cosets are disjoint, $g_{j_i}H \cap g_{k_i} = \emptyset$.

Then $gg_1H = g_{j_1}H$,..., $gg_nH = g_{j_n}H$ for $j_n \in \{1, ..., n\}$. Let $\sigma \in S_n$ be the permutation that maps 1 to j_1, 2 to j_2, ..., n to j_n. By the claim, $\{1, ..., n\}$ map to distinct numbers in $\{1, ..., n\}$, so σ is a well-defined permutation.

Define a homomorphism $\phi : G \longrightarrow S_n$ by $\phi(g) = \sigma_g$, where σ_g is defined as above.

To find ker ϕ:

$$\begin{aligned} x \in \text{ ker } \phi &\Longleftrightarrow xg_iH = g_iH \ \forall i \\ &\Longleftrightarrow g_i^{-1}xg_iH = H \ \forall i \\ &\Longleftrightarrow g_i^{-1}xg_i \in H \ \forall i \end{aligned}$$

In particular, $e^{-1}xe = x \in H$ for the coset H, so if $x \in$ ker ϕ, $x \in H$. Therefore, ker $\phi \leq H$.

But by the first isomorphism theorem, $G/\text{ker } \phi \cong \text{im } \phi \leq S_n$, so $|G|/|\text{ker } \phi| \leq |S_n| = n!$. Then if we let $N = \text{ker } \phi$, N is a normal subgroup contained in H so that the index of N in G is finite.

b) Let $(G : H_1) = a$ and let $(G : H_2) = b$. We must show that $|G|/|H_1 \cap H_2|$ is finite.

Claim 1: Let $K \leq H \leq G$. Suppose $(G : H) = n < \infty$. Then $(G : K) = (G : H)(H : K)$. (By the way, this is true without the second statement, but we'll prove it with the second

statement)
Proof of claim 1: First, suppose G is finite. Then by Lagrange's theorem, $(G:K) = |G|/|K|$, $(G:H) = |G|/|H|$, and $(H:K) = |H|/|K|$. Then $(G:H)(H:K)$= $|G|/|H| * |H|/|K| = |G|/|K| = (G:K)$.
Next, suppose G is not finite.
Case 1:$(H:K) = \infty$
If $(H:K) = \infty$, then there exists distinct cosets $x_1K, x_2K, ...$ of K in H. Then $x_1K, x_2K, ...$ are also distinct cosets of of K in G, so $(G:K) = \infty$.
Case 2: $(H:K) = m \leq \infty$ Claim 1a: If $(H:K) = m \leq \infty$, then if $x_1K, x_2K, ...x_mK$ are the cosets of K in H and $y_1H, ..., y_nH$ are the cosets of H in G, then we claim that $x_1y_1K, .., x_1y_nK, ...x_my_1K, ..., x_my_nK$ are the cosets of K in G.
Proof of Claim 1a:
Suppose x_ay_bK and x_cy_dK are cosets that are not disjoint. Then there exists k, k' such that $x_ay_bk = x_cy_dk'$, so x_aH and x_cH are not disjoint. This is a contradiction, so all of the cosets $x_1y_1K, .., x_1y_nK, ...x_my_1K, ..., x_my_nK$ are pairwise disjoint.
Next, suppose $x \in G$. Then since $x \in y_iH$ for some i, so $x = y_ih$ for some $h \in H$. Since the cosets $x_1K, x_2K, ...x_mK$ partition H, $h = x_jk$ for some j and $k \in K$, so $x = y_ix_jk \in y_ix_jK$. Therefore, $x_1y_1K, .., x_1y_nK, ...x_my_1K, ..., x_my_nK$ partition G.
Therefore, there are $(G:H)(H:K))$ cosets of K in G, so $(G:K) = (G:H)(H:K)$.

Since $H_1 \cap H_2 \leq H1$, and $(G:H_1)$ is finite, by claim 1, $(G:H_1 \cap H_2) = (G:H_1)(H:H_1 \cap H_2)$. By Proposition 3.13 in Dummit and Foote, $|H_1|/|H_1 \cap H_2| = |H_1H_2|/|H_2| \leq |G|/|H_2| = b \leq \infty$. Thus,

$$\begin{aligned}
|G|/|H_1 \cap H_2| &= (G:H_1)(H:H_1 \cap H_2) \\
&= a(H:H_1 \cap H_2) \\
&\leq ab < \infty
\end{aligned}$$

Solution to Problem 8
Let G be a group of order 56. $56 = 2^3 * 7$, so by the Sylow Theorems, G has a at least one Sylow-2 subgroup and at least one Sylow-7 subgroup.

By the Sylow theorems, $n_2 \equiv 1 \mod 2$ and $n_2|7$, so the only possibilities for n_2 are 1 or 7.
Also, $n_7 \equiv 1 \mod 7$ and $n_7|8$, so the only possibilities for n_2 are 1 or 8.

If $n_1 = 1$ or $n_7 = 1$, then by the Sylow theorems, unique Sylow theorems are normal in G, so G would have a normal Sylow-p subgroup. So suppose, by contradiction, that $n_2 = 7$ and $n_7 = 8$.

Let $Q_1, .., Q_8$ be the Sylow-7 subgroups. Then for any distinct Sylow-7 subgroups Q_i, Q_j, $Q_i \cap Q_j$ is trivial: $Q_i \cap Q_j \leq Q_i$, and by Lagrange's Theorem, $|Q_i \cap Q_j|$ divides $|Q_i|$. The order of Q_i is 7, so $|Q_i \cap Q_j| = 1$ or 7. But $Q_i \neq Q_j$, so $|Q_i \cap Q_j| \neq 7$, so $|Q_i \cap Q_j| = 1$.

Then the Q_i's intersect trivially pairwise. By Lagrange's Theorem, each non-identity element of a Sylow-7 subgroup has order 7, so each of the 8 Sylow-7 subgroups contribute 6 distinct elements of order 7 to G, so G has at least 48 distinct elements of order 7.

We have also assumed that there are at least 7 distinct Sylow-2 subgroups, say, $P_1, ..., P_7$. Each of the Sylow 2-subgroups has 8 elements, and in order for $P_1 \neq P_2$, there is at least

one element in P_2 that is not in P_1. Then $P_1 \cup P_2$ has at least 9 distinct elements, none of which are of order 7 by Lagrange's theorem.

Then G has at least 48 elements of order 7 and at least an additional 9 elements, so G has at least 57 elements. This is a contradiction, since G has order 56.

Therefore, $n_1 = 1$ or $n_7 = 1$, so by the Sylow theorems, the unique Sylow p-subgroup is a normal p-subgroup.

Solution to Problem 9

a) The semi-direct product of $(\mathbb{Z}/5)^5$ and S_5 is $\{(x, y) | x \in (\mathbb{Z}/5)^5, y \in S_5\}$, so the order of the semidirect product of $(\mathbb{Z}/5)^5$ and S_5 is equal to $|S_5| * |(\mathbb{Z}/5)^5| = 5! * 5^5 = 2^3 * 3 * 5^6 = 375000$.

b) Since $|(\mathbb{Z}/5)^5 \rtimes S_5| = 2^3 * 3 * 5^6$, any Sylow 5-subgroup has order 5^6.

$(\mathbb{Z}/5)^5 \rtimes \langle(12345)\rangle$ is $\{(x, y) | x \in (\mathbb{Z}/5)^5, \langle(12345)\rangle\}$, so it has $5^5 * 5 = 5^6$ elements, so $(\mathbb{Z}/5)^5 \rtimes \langle(12345)\rangle$ is a Sylow 5-subgroup. Generalizing this, if (abcde) is any permuatation in S_5, then $(\mathbb{Z}/5)^5 \rtimes \langle(abcde)\rangle$ is a Sylow 5-subgroup. Since $\langle(abcde)\rangle$ has order 5, $\langle(abcde)\rangle$ is a subgroup of S_5 with 5 elements. S_5 has $5!/5 = 4!$ 5 cycles, and each distinct subgroup of S_5 is a cyclic subgroup with 4 elements of order 5, so S_5 has $4!/4 = 6$ distinct subgroups of order 5, so there are 6 Sylow 5-subgroups in S_5.

To verify this, by the Sylow theorems, $n_5 \equiv 1 \mod 5$ and $n_5 | 24$, so the possibilities for n_5 are 1 or 6, and there is more than one Sylow 5-subgroup (from the previous paragraph), so there are 6 Sylow 5-subgroups.

Solution to Problem 11

Let G be a finite group and let H be a p-subgroup of G. Consider the action of H on G/H by multiplication on the left. By the Orbit-Stabilizer theorem,

$$|G/H| = \sum_i |H|/|Stab(g_i H)|$$

where i ranges over the coset representations in G/H.

Claim: $gH \in N_G(H)/H$ if and only if $|H|/|\text{Stab}(gH)| = 1$.
Proof of claim:

$$\begin{aligned} gH \in N_G(H)/H &\iff gHg^{-1} = H \\ &\iff g^{-1}hg \in H \,\forall\, h \in H \\ &\iff g^{-1}hgH = H \,\forall\, h \in H \\ &\iff hgH = gH \,\forall\, h \in H \\ &\iff \text{ Stab } (gH) = H \\ &\iff |H|/|\text{ Stab } (gH)| = 1 \end{aligned}$$

Then

$$|G/H| = [N_G(H) : H] + \sum_j |H|/|\text{Stab}(g_j H)|$$

where j ranges over all $g_j H$ such that the its stabilizer is nontrivial. Then

$$|G/H| - [N_G(H) : H] = \sum_j |H|/|\text{Stab}(g_j H)|$$

But since H is a p-subgroup, $|H|/|\text{Stab}(g_j H)|$ is a power of p, and Stab $(g_j H)$ is nontrivial for all j, so p divides $\sum_j |H|/|\text{Stab}(g_j H)|$. Therefore,

$$\begin{aligned} |G/H| - [N_G(H) : H] &\equiv \sum_j |H|/|\text{Stab}(g_j H)| (\text{mod p}) \\ &\equiv 0 \ (\text{mod p}) \end{aligned}$$

since $|H|/|Stab(g_j H)|$ are all divisible by p. Therefore,

$$|G/H| \equiv [N_G(H) : H] \ (\text{mod p})$$