November 3rd, 2009

MAT240: Abstract Linear Algebra Lecture:

We have an isomorphism:

(abstract, coordinate-free) → (practical, basis-dependent)

$L(V, M) \overset{\varphi}{\Rightarrow} M_{mxn}(F)$

$T \to [T]_{\beta}^{\gamma^n} = A$

$$A = \left([T_{(v_1)}]_\gamma | [T_{(v_2)}]_\gamma \dots | [T_{(v_n)}]_\gamma\right) = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & A_{ij} & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

(* $W \ni T_{(v_j)} = \sum_{i=1}^{m} A_{ij} w_i$

$1 \le j \le n$)

φ is one-to-one and onto:

- Output determines input ↔ given A, you can recover T ↔ True ∴ φ is one-to-one.
- Onto ↔ given a matrix A (of correct dimensions) we can find T. Given A, construct T using (*), (remember that a linear transformation I can be assigned arbitrary values on a basis of its domain. By construction, the matrix associated with T is A.

Definition Proposition: $L(V, W)$ is a vector space with operations as follows:

$T, S \in L(V, W)$ let $(T + S) \in L(V, W)$ be given by $(T + S)(U \in V) = T(U) + S(U)$

$c \in F(cT)(u) \coloneqq c\big(T(u)\big) \to 0_{L(V,W)}(U) = 0$

Part of Proof:

Claim 1: T+S really is a linear transformation

$\to (T + S)(U + V) = T(U + V) + S(U + V) = T(U) + T(V) + S(U) + S(V)$

$\to (T + S)(U) + (T + S)(V) = T(U) + S(U) + T(V) + S(V)$

Likewise $(T+S)(cU) = c(T+S)U$.

Claim 2: cT is a linear transformation:

… (as previous)

Claim 3: $0_{L(V,W)}$ is a linear transformation

Next: Verify Commutativity and Associativity

To show that $T \to [T]_\beta^\gamma$ is an isomorphism of VS we just need to show:

1. $[T+S]_\beta^\gamma = [T]_\beta^\gamma + [S]_\beta^\gamma$
2. $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ (Similar to below)

Proof of 1:

$C = [T+S]_\beta^\gamma,\ \ A = [T]_\beta^\gamma,\ \ B = [S]_\beta^\gamma$ (need C=A+B)

$W \ni \sum_{i=1}^m c_{ij} w_i = (T+S)v_j = T(v_j) + S(v_j)$

$= \sum A_{ij} w_i + \sum B_{ij} w_i = \sum (A_{ij} + B_{ij}) w_i$

$\to \forall i \ \ c_{ij} = A_{ij} + B_{ij} \to C = A + B$ ■

Idea: $L(V,W) \leftrightarrow$ Matrices on L there's a composition T, S, T of S should be a product on matrices.

$$U(of\ \dim(P)) \xRightarrow{S\ (B=[S]_\gamma^\beta)} V(of\ \dim(n)) \xRightarrow{T\ (A=[T]_\beta^\gamma)} W\ (of\ \dim(m))$$

$$\xRightarrow{T\ of\ S\ (C=[TofS]_\alpha^\gamma)}$$

Challenge: Given A&B, find a formula for C.

A is the matrix with $T(v_j) = \sum_{j=1}^m A_{ij} w_i \quad A \in M_{mxn}$

B “” $S(u_k) = \sum_{j=1}^n B_{jk} v_j \quad B \in M_{nxp}$

C “” $(Tof\ S)(u_k) = \sum_{i=1}^m c_{ik} w_i \quad C \in M_{mxp}$

$$= T(\sum_{j=1}^{n} B_{jk} v_j) = \sum_{j=1}^{n} B_{jk} T(v_j) = \sum_{j=1}^{n} B_{jk} \sum_{i=1}^{m} A_{ij} w_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} B_{jk} A_{ij} w_i$$

$$= \sum_{i=1}^{m} (\sum_{j=1}^{n} A_{ij} B_{jk}) w_i$$

$$\to \sum_{i=1}^{m} C_{ik} w_i \quad \therefore C_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$$

Definition: Given $A \in M_{mxn}$, $B \in M_{nxp}$ let $AB \in M_{mxp}$ be given by: $(AB)_{ik} \equiv \sum_{j=1}^{n} A_{ij} B_{jk}$

Theorem: $U \overset{S}{\to} V \overset{T}{\to} W \quad [T\ of\ S]_{\alpha}^{\gamma} = [T]_{\beta}^{\gamma}\ of\ [S]_{\alpha}^{\beta}$ ■

Example 1:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

$$C = AB \in M_{2x3}$$

$$C = \begin{pmatrix} -2 & 5 & 12 \\ -2 & 11 & 30 \end{pmatrix}$$

$$c_{11} = \sum_{j=1}^{3} A_{ij} B_{ji} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$$

$$c_{23} = \sum_{j=1}^{3} A_{ij} B_{ji} = A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33}$$

BxA We must multiply an mxn matrix by an nxp matrix.

Example 2:

For $A = \begin{pmatrix} cos\alpha & -sin\beta \\ sin\alpha & cos\alpha \end{pmatrix}$

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Example 3:

$A \in M_{mxn}$ define a linear transformation $T_A: F^n \to F^m$ by:

$$v \in F^n = M_{nx\,1} \to Av \in M_{mx\,1} = F^m$$

(Easy to check that T_A is a linear transformation)

$$[T_A]_{ej}^{ei} = A \quad \text{(proof to follow)}$$