

Homework 7. NOT HAND IN.

Section 11.

1. Let $A = Q \cap [0, 1]$ $A \subset \mathbb{R}^1$.

A is measure-0 in \mathbb{R}^1 since we can take $Q = \{q_i\}$ be a listing of elements in A .

Then let $R_i = [q_i - \frac{0.96}{2^{i+1}}, q_i + \frac{0.96}{2^{i+1}}]$ s.t. $\cup R_i \supset A$.

and $\cup R_i = \sum_{i=1}^{\infty} \frac{0.96}{2^i} = 0.96 \in \epsilon \Rightarrow A$ is measure-0.

$\bar{A} = [0, 1]$ is an interval $\Rightarrow \bar{A}$ is not measure-0.

But $A = [0, 1]$ is an interval $\Rightarrow B \setminus A$ is not measure-0.

3. Let R_i be rectangles that ^{their union} cover \mathbb{R}^{n-1} s.t. $\cup R_i \supset \mathbb{R}^{n-1}$ $\sum_{i=1}^{\infty} V(R_i) > V(\mathbb{R}^{n-1})$

Let $R_i' = R_i \times [r, r]$ s.t. $r = \frac{\epsilon}{2 \sum_{i=1}^{\infty} V(R_i)}$ given $\epsilon > 0$.

so $\cup R_i'$ covers $\mathbb{R}^n \times 0$. & $\sum_{i=1}^{\infty} V(R_i') = \sum_{i=1}^{\infty} V(R_i) \epsilon r = 2r \sum_{i=1}^{\infty} V(R_i)$

$$= 2 \cdot \frac{\epsilon}{2 \sum_{i=1}^{\infty} V(R_i)} \cdot \sum_{i=1}^{\infty} V(R_i) = \epsilon < \epsilon.$$

Hence $\mathbb{R}^n \times 0$ has measure-0 in \mathbb{R}^n .

5. Since A is measure-0 in \mathbb{R}^n so there is a covering Q_1, Q_2, \dots of A by countably many rectangles s.t. $\sum_{i=1}^{\infty} V(Q_i) < \epsilon$.

Since A is compact, so $\exists K$ s.t. Q_1, Q_2, \dots, Q_k covers A & $\sum_{i=1}^k V(Q_i) < \epsilon$.

& $\sum_{i=1}^k V(Q_i) < \sum_{i=1}^{\infty} V(Q_i) < \epsilon$ as required.

7. (i) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ when $x \in [0, 1]$ f is dis-contin. at every point in $[0, 1]$ since there \exists an irrational # between every 2 \mathbb{Q} # & there \exists an \mathbb{Q} # between every 2 irrational #.

(ii) $x \in \mathbb{Q}$, consider the sequence $x_n = x + \frac{\sqrt{2}}{n}$. we have $(x_n) \rightarrow x$.

yet x_n is irrational $\forall n$.

since if $x_n \in \mathbb{Q}$, then $x_n - x \in \mathbb{Q}$ then $n(x_n - x) = \sqrt{2} \in \mathbb{Q} \Rightarrow \epsilon$.

(iii) $x \notin \mathbb{Q}$, suppose $x = m + 0.d_1 d_2 \dots$ where $m \in \mathbb{Z}$.

Let $x_n = m + 0.d_1 d_2 \dots d_n$ (i.e) x_n is the decimal representation of x cut off at the n^{th} digit after the decimal point ($x_n = m$) then every $x_n \in \mathbb{Q}$ & $\lim_{n \rightarrow \infty} x_n = x$.

(ii) $f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ in } [0, 1] \\ 0 & \text{otherwise} \end{cases}$ f is dis-contin. at \mathbb{Q} since f is cont. when $x \notin \mathbb{Q}$ $f(x) = 0$.

since $f(x) = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ when $x \in \mathbb{Q}$, so there are at most $n!$ many x s.t. $f(x) \geq \frac{1}{n}$.

since there are finite many of them, so we can pick $\delta > 0$ s.t. $\forall x \in B_\delta(x_0), f(x) < \frac{1}{n}$.

$\delta = \min\{\delta_1(x_0, x_1), \delta_2(x_0, x_2), \dots\}$ so f is dis-contin. on at most $\mathbb{Q} \cap [0, 1]$ as measure-0.

So f is integrable on $[0, 1]$ since $f = 0$ almost everywhere. $\int f = 0$.

10 Let \mathcal{P} be a partition of $\bigcup_{k=1}^{\infty} Q_k$ s.t. P_i is defined by the end points of its component of Q, Q_1, Q_2, \dots & $\mathcal{P} = \{P_1, \dots, P_n\}$. ~~Assume~~ (assume $Q \subset \mathbb{R}^m$)

Then each rectangle Q, Q_1, Q_2, \dots is a union of subrectangles determined by \mathcal{P} .

$$\text{So } v(Q) = \sum_{R \in \mathcal{P}} v(R).$$

Since Q is covered by $\bigcup_{k=1}^{\infty} Q_k$, so every $R \subset Q$ must be at least in one Q_k .

$$\text{So } \sum_{R \in \mathcal{P}} v(R) \leq \sum_{k=1}^{\infty} \sum_{R \subset Q_k} v(R) = \sum_{k=1}^{\infty} v(Q_k) \text{ by HM 124. } \Rightarrow v(Q) \leq \sum v(Q_k) \text{ as required}$$