Define the matrix $[A]$ to be the $n \times |J|$ matrix whose η^{th} column is $A(e_{x_\eta}) = x_\eta \in X \subset R^n$.

(In the case where $J = \{1, 2, \dots\}$, $[A] = \left(A(e_{x_1}), A(e_{x_2}), \dots, A(e_{x_j}), \dots \right)$.)

Then, by extending the def^n of matrix mult, we have that $A(a) = [A] \begin{pmatrix} \text{the } |J| \times 1 \text{ column} \\ \text{vector whose} \\ \eta^{th} \text{ entry is } a(x_\eta) \end{pmatrix}$

In the case where $J = \{1, 2, \dots\}$,

$$A(a) = [A] \begin{pmatrix} a(x_1) \\ a(x_2) \\ \vdots \end{pmatrix} = \left(A(e_1), A(e_2), \dots \right) \begin{pmatrix} a(x_1) \\ a(x_2) \\ \vdots \end{pmatrix} = \sum_{j=1}^{\infty} a(x_j) A(e_j) \left(= A(a) \right)$$

Aside: This is by analogy to linear algebra: If $T : F^k \to F^n$, linear trans.,

$T(x) = T\left(\sum_{j=1}^{k} x_j e_j \right) = \sum_{j=1}^{k} x_j T(e_j)$, so if $[T] = \left(T(e_1), T(e_2), \dots, T(e_k) \right) \in M_{n \times k}(F)$, then

$$T(x) = [T] \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

So, the R-mod. morphism A can be interpreted as an $n \times |J|$ matrix $[A]$ whose η^{th} column is $A(e_\eta)$.
(acting by multiplication on $|J| \times 1$ column vectors in R of which only finitely many entries are nonzero)

Similarly to linear algebra, we only need to know where the $\underbrace{\text{"basis" vectors of } X}_{e_{x_\eta}\text{'s}}$ are mapped by A. !!!
where elements of R^X are "coordinate vectors" w.r.t the e_{x_η}'s, i.e. the η^{th} coord. of a is $a(x_\eta)$. $\left(a = \sum_{\eta \in J} a(x_\eta) e_{x_\eta} \right)$

Notice: $\text{im}(A) = \text{span}_R\left(\{ A(e_{x_\eta}) : \eta \in J \} \right) = \text{span}_R(\text{columns of } [A])$.

Summary: If M is a module gen. by n elements, $M = \langle g_1, \dots, g_n \rangle$, $X = \ker(\pi) = \langle x_\eta : \eta \in J \rangle = \text{im}(A)$

$$R^X \xrightarrow[a \mapsto \sum_{\eta \in J} a(x_\eta) x_\eta]{A} R^n \xrightarrow[\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mapsto \sum_{i=1}^{n} r_i g_i]{\pi} M,$$ then $X = \text{im}(A)$; $A(e_{x_\eta}) = x_\eta = \text{col}_\eta([A])$.

So, every $\underset{\in M_R(n \times |J|)}{n \times |J| \text{ matrix } A}$ defines a module generated by n elements, $M_A = R^n / \text{im}A \cong M$.

Note: It would have been better to define not R^X but $R^{\{x_\eta : \eta \in J\}}$ b/c that's all we really needed, and b/c that would have strengthened the analogy to linear algebra: Think of the $\{ e_{x_\eta} : \eta \in J \}$ as a basis for $R^{\{x_\eta : \eta \in J\}}$ and think of $\{ x_\eta : \eta \in J \} \subset X \subset R^n$ as a basis for X.
Recall: $\text{im}(A) = X$, so A is mapping a "basis" to a "basis"; A is analogous to a lin. trans.

(although for this analogy to be even better, need a notion of the x_η's as minimal generators)