## SUMMARY FOR THE ALGEBRA EXAM

## 1. Basic Notions

Theorem 1. $H \subset G$ is a subgroup iff $x, y \in H \Rightarrow x y^{-1} \in H$.
Theorem 2. If $f$ is an isomorphism, then $f^{-1}$ is an isomorphism.
Theorem 3. $N \leq G$ is normal $\Leftrightarrow g x g^{-1} \in N \forall x \in N, g \in G$.
Definition 1 (Conjugaison). In Prof. Bar-Natan's notation, $x^{g}=g^{-1} x g$. Automorphism $=$ self-isomorphism, and the function defined by

$$
x \rightarrow g x g^{-1}
$$

is the Inner automorphism coming from $g$.
Theorem 4. $\operatorname{Inn}(G) \triangleleft \operatorname{Aut}(G)$
Theorem 5. $\operatorname{Ker}(\phi) \triangleleft G$
Theorem 6. Suppose $N \triangleleft G$. Then there exists a group $H$ and an homomorphism $\phi: G \rightarrow H$ such that $N=\operatorname{Ker}(\phi)$.

Remark 1. Take $H=G / N$.
Definition 2 (Centralizer, normalizer, center). $X \subset G$.

- Centralizer: $C_{G}(X)=\left\{g \in G \mid g x g^{-1}=x \forall x \in X\right\}$.
- Centre: $Z(G)=C_{G}(G)$.
- Normalizer: $N_{G}(X)=\left\{g \in G \mid g X g^{-1}=X\right\}$.

Remark 2. If $X \leq G$, then X is a normal subgroup iff $N_{G}(X)=G$.
Theorem 7. Let $\langle X>$ be the subgroup of $G$ generated by $X$, i-e the smallest subgroup of $G$ containing $X$. Then $C_{G}(X)=C_{G}(<X>)$ and $N_{G}(X)=N_{G}(<$ $X>$ ).

## 2. Isomorphism Theorems

Theorem 8 (Fundamental theorem of Homomorphisms). Let $G, H$ be two groups and $\phi: G \rightarrow H$ be an homomorphism. Let $K \triangleleft G$ and $\psi: G \rightarrow G / K$ be the natural surjective homomorphism. If $K \subset \operatorname{Ker}(\phi)$, then there exists a unique homomorphism $\alpha: G / K \rightarrow H$ such that $\phi=\alpha \psi$.
Theorem 9 (First Isomorphism theorem). Let $\phi: G \rightarrow H$ be an homomorphism. Then $G / \operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$.
Theorem 10. $H, K \leq G$. Then $H K \leq G \Leftrightarrow H K=K H$.
Corollary 11. Let $H, K$ be subgroups of $G$. If $H \subset N_{G}(K)$, then $H K \leq G$ and $K \triangleleft H K$.

Corollary 12. If $K \triangleleft G$, then $H K<G$ for any $H \leq G$.

Theorem 13 (Second Isomorphism Theorem). Let $H, K$ be subgroups of $G$ such that $H \subset N_{G}(K)$. Then, $(H \cap K) \triangleleft H, K \triangleleft H K$ and $H K / K \cong H /(H \cap K)$.

Theorem 14 (Third Isomorphism Theorem). Let $K \triangleleft G$, $H \triangleleft G$, $K \leq H$. Then, $H / K \triangleleft G / K$ and $(G / K) /(H / K) \cong G / H$.

Theorem 15. Let $N \triangleleft G$. Then $q: G \rightarrow G / N$ induces a bijection between the subgroups of $G$ which contain $N$ and the subgroups of $G / N$.

## 3. Jordan- Holder Theorem

Theorem 16. Givin a finite $G$, there exists a sequence $G=G_{0} \triangleright G_{1} \triangleright G_{2} \triangleright G_{3} \triangleright \ldots \triangleright$ $G_{n}=\{e\}$ such that $H_{i}=G_{i} / G_{i+1}$ is simple. Furthermore, the set $\left\{H_{0}, \ldots, H_{n-1}\right.$ is unique up to a permutation.

Remark 3. The sequence is called a tower and the set is called a composition series of $G$.

Definition 3 (solvable groups). A solvable group is a group whose Jordan-Holder composition series has only Abelian factors.

Definition 4 (commutator). Given $x, y \in G$, the commutator $[x, y]$ is given by $[x, y]=x y x^{-1} y^{-1}$. We denote by $G^{\prime}$ the group generated by all commutators of $G$.

Theorem 17. $G^{\prime} \triangleleft G, G / G^{\prime}$ is abelian and any morphism from $G$ into an abelian group factors through $G / G^{\prime}$.

Remark 4. A equivalent definition of solvable groups is the following: A group $G$ is solvable is there exists an $n$ such that $G^{(n)}=\{e\}$, where $G^{(n)}=\left(G^{(n-1)}\right)^{\prime}$. The two definitions are equivalent because for every group $G$ and every normal subgroup $N$ of $G$, the quotient $G / N$ is abelian $\Leftrightarrow N$ contains $G^{\prime}$.

Example 1. $A_{n}$ is not solvable for $n \geq 5$, because they are simple. In particular, $\left[A_{n}, A_{n}\right]=A_{n}$.
Theorem 18. If $N \triangleleft G$, then $G$ is solvable $\Leftrightarrow N$ and $G / N$ are. Also, if $H<G$ and $G$ is solvable, then $H$ is solvable.

## 4. Symmetric Group

Theorem 19. Let $\sigma, \tau \in S_{n}, \sigma=\left(a_{1}^{(1)} \ldots a_{1}^{\left(r_{1}\right)}\right) \ldots\left(a_{n}^{(1)} \ldots a_{n}^{\left(r_{n}\right)}\right)$. Then,

$$
\tau \sigma \tau^{-1}=\left(\tau^{-1}\left(a_{1}^{(1)}\right) \ldots \tau^{-1}\left(a_{1}^{\left(r_{1}\right)}\right)\right) \ldots\left(\tau^{-1}\left(a_{n}^{(1)}\right) \ldots \tau^{-1}\left(a_{n}^{\left(r_{n}\right)}\right)\right)
$$

Corollary 20. $\tau \sigma \tau^{-1}$ have the same cycle decomposition as $\sigma$. Also, $\sigma$ is conjugated to $\sigma^{\prime}$ if and only if $\sigma$ and $\sigma^{\prime}$ have the same cycle type.

Corollary 21. The number of conjugacy classes in $S_{n}$ is equal to the number of partitions of $n$.

Theorem 22. $A_{n}$ is simple.
Theorem 23. $S_{4}$ contains no normal subgroup isomorphic to $S_{3}$.

## 5. Group action on Sets

Definition 5 (Group action). A group $G$ acting on a set $X$ is a binary map $G \times X \rightarrow X$, denoted $(g, x) \mapsto g x$ such that

- $\left(g_{1}, g_{2}\right) x=g_{1}\left(g_{2} x\right)$
- $e x=x$

Then $X$ is called a $G$-set.
Definition 6 (Transitive set, Stabilizer, Orbits, fix points). Four definitions:

- Transitive set: $X$ is transitive if for all $x, y \in X$, there exists $g \in G$ such that $g x=y$.
- Orbits: $\operatorname{Orb}(x)=G x=\{g x \mid g \in G\} . X / G$ is the set of all orbits.
- Stabilizer: $\operatorname{Stab}_{X}(x)=G_{x}=\{g \in G \mid g x=x\}<G$.
- Fix Points : $X^{g}=\{x \in X \mid g x=x\}$.

Theorem 24. Three properties of group actions:

- Every $G$-set $X$ is a disjoint union of transitive $G$-sets. Those are the orbits.
- If $Y$ is a transitive $G$-set, then $Y \cong G / H$ as a $G$-set for some $H<G$.
- If $X$ is a transitive $G$-set and $x \in X$, then $X \cong G /\left(\operatorname{stab}_{X}(x)\right.$. So, $|X|| | G \mid$.

Corollary 25. If $|X|<\infty$ and $x_{i}$ are representatives of the orbits, then

$$
|X|=\sum_{i} \frac{|G|}{\left|\operatorname{Stab}_{X}\left(x_{i}\right)\right|}=\sum_{i} G_{x_{i}}
$$

Theorem 26.

$$
|G x|=\left[G: G_{x}\right]
$$

Theorem 27 (Burnside's lemma).

$$
|X / G|=\frac{1}{|G|} \sum_{g \in G}\left|X^{g}\right|
$$

Theorem 28. If $|X|<\infty$ and $x_{i}$ are the representatives of the orbits with more then one element, then

$$
|G|=|Z(G)|+\sum_{i} \frac{|G|}{\left|C_{G}\left(x_{i}\right)\right|}
$$

6. SYLOW'S THEOREMS

Definition 7 (Sylow-p subgroup). Let $|G|=p^{\alpha} m$, with $p \nmid m$. We say that $P<G$ is a Sylow-p subgroup of $G$ if $|P|=p^{\alpha}$.
Lemma 29 (Cauchy Theorem). If $A$ is an abelian group and $p||A|$, then there exists some element of order $p$ in $A$.

Theorem 30 (Sylow Theorem 1). Let $\operatorname{Syl}_{p}(G)=\left\{P<G:|P|=p^{\alpha}\right\}$. Then $\operatorname{Syl}_{p}(G) \neq \emptyset$.
Theorem 31 (Sylow Theorem 2). Every p-subgroups of $G$ is contained in some Sylow-p subgroup $G$.

Two lemmas for :
Lemma 32. If $P \in \operatorname{Syl}_{p}(G)$ and $H<N_{G}(P)$ is a p-group, then $H<P$.

Lemma 33. If $x \in G$ has order a power of $p$, and $x \in N_{G}(P)$, then $x \in P$.
Theorem 34 (Sylow Theorem 3). Let $n_{p}(G)=\left|S y l_{p}(G)\right|$. Then,

- $n_{p}| | G \mid$;
- $n_{p} \equiv 1 \bmod p$;
- All Sylow-p subgroups of $G$ are conjugate (so isomorphic) to each other.

The Sylow-1 and Sylow-2 theorems can be deduced from the two following propositions.

Proposition 35. If $R \in S y l_{p}(G)$, then $n_{R}(G)=\mid$ conjugates of $R \mid \equiv 1 \operatorname{modp}$ and $n_{R}(G)| | G \mid$.
Proposition 36. If $H$ is a p-subgroup of $G$ and $P \in \operatorname{Syl}_{p}(G)$, then $H$ is contained in a conjugate of $P$.

Example 2. See the example in the notes of groups of order $12,15,21$.

## 7. SEmidirect Product

Theorem 37. If $K \triangleleft G, H \triangleleft G, K \cap H=\{e\}$, then $K H \cong K \times H$.
Definition 8 (Semidirect product). Suppose $N, H$ are groups and $\phi: H \rightarrow \operatorname{Aut}(N)$ is a homomorphism. Then the semi-direct product of $N$ and $H$ is defined as follow:

$$
N \rtimes_{\phi} H=N \times H
$$

with product

$$
\left(n_{1}, h_{1}\right)\left(n_{2}, h_{2}\right) \equiv\left(n_{1} \phi\left(n_{2}\right), h_{1} h_{2}\right)
$$

Theorem 38. $G=N \rtimes H$ is a group. Also, $H<G, N \triangleleft G, G / N=H, G=N H$.
Theorem 39. If $G=N H, N \triangleleft G, H<G, H \cap N=\{e\}$. Then, $G \cong N \rtimes_{\phi} H$, where $\phi_{h}(n)=h n h^{-1}$.

## 8. SOME USEFUL FACTS IN THE HOMEWORK

Theorem 40. Let $G$ be a group. Then, $\phi: G \rightarrow G$ given by $\phi(g)=g^{2}$ is a homomorphism if and only if $G$ is abelian.

Theorem 41. If $G / Z(G)$ is cyclic, then $G$ is abelian. Moreover, if $\operatorname{Aut}(G)$ is cyclic, then $G$ is abelian.
Theorem 42. Let $G$ be a group, and $H$ be a subgroup of finite index. Then there exists a normal subgroup $N$ of $G$ such that $N$ is contained in $H$ and $[G: N]<\infty$.

