MAT1100HF: ALGEBRA STUDYING

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Abstract. This is a set of study notes for MAT1100HF from the University of Toronto

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Summary of Proofs on Rings

In the following, let R be a commutative ring.

Theorem. *R/I is a field if and only if I is maximal*

Proof idea: (Forward) A field F only has ideals 0 and F because every element is a unit. Apply ISO 4. (Backward) ISO 4.

Theorem. *Every ideal is contained in a maximal ideal*

Proof idea: Pick an arbitrary ideal I and consider the partially ordered set of all proper ideals in which I is contained. For a given chain of increasing ideals, take the union of all ideals in the chain. Show that this is an ideal properly contained in R. Invoke Zorn's lemma.

Proposition. *P is a prime ideal if and only if R/P is an integral domain.*

Proof: (Forward) Suppose that $[a][b] = [ab] = 0$ then $ab \in P \Rightarrow a \in P$ or $b \in P$. So $[a] = 0$ or $[b] = 0$. (Backward) $[ab] = [a][b] = 0 \Rightarrow [a] = 0$ or $[b] = 0$; thus $ab \in P \Rightarrow a \in P$ or $b \in P$.

Proposition. *Every maximal ideal is prime*

Proof idea: Apply the previous propositions. A field is, in particular, an integral domain.

Proposition. *In an integral domain, every prime element is irreducible*

Proof idea: Use the definition of prime and divisibility

Proposition. *In an integral domain, every prime factorization is unique*

Proof idea: Assume distinct prime factorizations and use divisibility arguments to eliminate primes from the factorization one by one.

Proposition. *In a unique factorization domain, an element is prime if and only if it is irreducible*

Proof idea: (Forward) A UFD is an integral domain. (Backward) Suppose an irredicuble element has a factorization into more than one prime, apply definition of irreducibility.

Proposition. *R is a UFD if and only if every element has a unique factorization into irreducibles*

Proof idea: (Forward) In a UFD there is always a unique factorization into primes and every prime is irreducible. (Backward) Show that an irreducible element must be prime. Suppose that x irreducible and $x|ab$ for some a and b. Decompose everything into irreducibles, then use uniqueness to show that x is associate to one of the irreducible factors of a or b.

Proposition. *In a UFD, the GCD of two elements always exists*

Proof idea: Take the prime factorizations of two elements and pick the minimal power of each prime.

Theorem. *Every Euclidian domain is a PID*

Proof idea: Find an element x of the ideal I with minimal norm, and show two sided containment to get $I = \langle x \rangle$. One direction follows trivially since products of x are in I. The other direction follows by picking any element in y and using the Euclidian algorithm so $y = nx + r$. Either $r = 0$ (done) or r contradicts minimalizty of x.

Theorem. *Every PID is Noetherian*

Proof idea: The union of an infinite chain of ideals is generated by some element x. Then, x is in some ideal in the chain and thus every subsequent ideal.

Theorem. *Every prime ideal in a PID is maximal*

Proof idea: Suppose the existence of an ideal containing $\langle x \rangle \supseteq \langle p \rangle$ then $p = ax \Rightarrow p|ax$. Since p is prime either $p|a$ or $p|x$. In the former case, show that x is a unit and $\langle x \rangle = R$. In the latter case, $x = np$ implies that the ideal generated by p contains the ideal generated by x contradicting maximality.

Theorem. *Every principal ideal domain is a unique factorization domain*

Proof idea: For some element x find the maximal (prime) ideal containing $\langle x \rangle$ then $x = p_1 x_2$ for some x_2. Continue inductively. The process must terminate because a PID is Noetherian.

Proposition. *In a PID, $\langle a, b \rangle = \langle \gcd(a, b) \rangle$; $\gcd(a, b) = sa + tb$ for some $s, t \in R$.*

Proof idea: Suppose the existence of a principal generator q for the ideal and show that any other element dividing both a and b must divide q.