LECTURE NOTES FOR OCTOBER 27

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Definition 0.1. (1) A ring R is a set together with two binary operations $+$ and $\times$ (called addition and multiplication) satisfying the following axioms:
(i) $(R, +)$ is an abelian group
(ii) $\times$ is associative
(iii) the distributive laws hold in R. That is, for all $a, b, c \in R$

$$(a + b) \times c = (a \times c) + (b \times c) \text{ , and } a \times (b + c) = (a \times b) + (a \times c)$$

(2) The ring R is commutative if multiplication is commutative.
(3) The ring R is said to have an identity if there is an element $1 \in R$ with

$$1 \times a = a \times 1 = a, \forall a \in R$$

As an aside, consider a different structure called a quandle.

Definition 0.2. A quandle is a set Q with a binary operation $*$ such that:
(1) For all $a, b, c \in Q$, $a * (b * c) = (a * b) * (a * c)$
(2) For all $a, b \in Q$, there exists a unique $c \in Q$, such that $a * c = b$
(3) For all $a \in Q$, $a * a = a$

If G is a group , the $(G, *)$ where $*$ is conjugation, is a quandle.

Lemma 0.3. *If we denote the additive identity of a ring R by 0, then $0 \times a = 0 = a \times 0$ for all $a \in R$.*

Common examples of rings include $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. For any ring R we can form the ring of polynomials with coefficients in R, denoted by $R[x]$. The set of n by n matrices with entries in R (Denoted by $M_n(R)$) with the usual matrix multiplication and addition.

Definition 0.4. If R, S are rings, a ring homomorphism, $\phi : R \to S$ is a set map so that for all $a, b \in R$:
(1) $\phi(a + b) = \phi(a) + \phi(b)$
(2) $\phi(ab) = \phi(a)\phi(b)$

Theorem 0.5. *Rings together with ring homomorphisms forms a category*

Examples of homomorphisms:
(1) $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$
(2) $R \to R[x]$, $a \to ax^0$
(3) $R \to M_n(R)$, $a \to aI_n$
(4) $ev_u : R[x] \to R$, $p(x) \to p(u)$

In the last example, $p(x) \in R[x]$. Also, (4) is only a ring homomorphism if $u \in Z(R) = \{b : ba = ab \ , \forall a \in R\}$

Consider the set of n by n matrices with entries in $R[x]$ versus the set of polynomials with coefficients in $M_n(R)$. We can show that $M_n(R[x])$ and $(M_n(R))[x]$ are isomorphic rings. The maps can be illustrated by the following example:

$$\begin{bmatrix} 2x+7 & -3x \\ x^2-2 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 7 & 0 \\ -2 & 0 \end{bmatrix}$$

We can define a subring, and a kernal and image of a homomorphism in the natural way. Its not hard to show that the image of a homomorphism is a subring, however a kernal is not a subring - its an ideal (something which will be discussed further in the next class).