HOMEWORK ASSIGNMENT #2

CORRECTION

Algebra I

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Exercise 3.(a) page 6: *Find the equation of the plane containing the following points in space* $(2,-5,-1)$, $(0,4,6)$ *and* $(-3,7,1)$.

Let $A=(2,-5,-1), B=(0,4,6), C=(-3,7,1)$ and O be the origin. As proved on the text book page 5, if x belongs to the plane containing the points A, B, C, it has to verify :

$$x = A + s\cdot\overrightarrow{AB} + t\cdot\overrightarrow{AC}$$

where $s, t \in \mathbb{R}$.

Therefore we have:

$$x = A + s\cdot\left(\overrightarrow{AO}+\overrightarrow{OB}\right) + t\cdot\left(\overrightarrow{AO}+\overrightarrow{OC}\right) = A + s\cdot\left(\overrightarrow{OB}-\overrightarrow{OA}\right) + t\cdot\left(\overrightarrow{OC}-\overrightarrow{OA}\right)$$

So:

$$x = A + s\cdot(B-A) + t\cdot(C-A)$$

Moreover:

$$B-A=(0,4,6)-(2,-5,-1)=(-2,9,7)$$
$$C-A=(-3,7,1)-(2,-5,-1)=(-5,12,2)$$

So the equation is:

$$x=(2,-5,-1)+s\cdot(-2,9,7)+t\cdot(-5,12,2)$$

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Exercise 1 page 12-13: *Label the following statements as true or false.*

Statements		Label	Comments
(a)	Every vector space contains zero vector	True	Property VS3
(b)	A vector space may have more than one zero vector	False	Uniqueness of zero vector follows from cancellation law for vector addition
(c)	In any vector space, $ax = bx \Longrightarrow a = b$	False	Let $x = 0_V$ (zero vector). Suppose $a \neq b$, then $ax = 0_V = bx$ with $a \neq b$
(d)	In any vector space, $ax = ay \Longrightarrow x = y$	False	$V :=$ a vector space over $\mathbb{R}$, $a = 0 \in \mathbb{R}$, $x, y \in V$ s.t. $x \neq y \Longrightarrow ax = ay$ with $x \neq y$
(e)	A vector in F^n may me regarded as a matrix in $\mathcal{M}_{(n\times1)}(F)$	True	
(f)	An $m \times n$ matrix has m columns and n rows	False	It is the contrary: An $m \times n$ matrix has m rows and n columns
(g)	In $P(F)$, only polynomials of the same degree may be added	False	$P(F)$ is the set of all polynomials of any degree with coefficient from F. All of its elements can be added since $P(F)$ is a vector space.
(h)	if f and g are polynomials of degree n, then $f+g$ is a polynomial of degree n	False	Let f, g be polynomials of degree n such that $g = -f$ then $f + g = 0$ is not a polynomials of degree n
(i)	if f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n	True	$c\cdot(a_n x^n + \cdots + a_0) = (ca_n)x^n + \cdots + (ca_0)$ with $ca_n \neq 0$ since $c \neq 0 \wedge a_n \neq 0$
(j)	A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero	True	$c = c\cdot 1 = c\cdot x^0$ with $c \neq 0$
(k)	Two functions in $\mathcal{F}(S,F)$ are equal if and only if they have the same value at each element of S	True	Two functions f and g in $\mathcal{F}(S,F)$ are equal $\Longleftrightarrow$ $f(s) = g(s)\ \forall s \in S$

Exercise 7 page 14: *Let* $S = \{0,1\}$ *and* $F = \mathbb{R}$. *In* $\mathcal{F}(S,R)$ *show that* $f = g$ *and* $f + g = h$ *where* $f(t) = 2t + 1, g(t) = 1 + 4t - 2t^2$ *and* $h(t) = 5^t + 1$

Let $S = \{0,1\}$ and $F = \mathbb{R}$. Two functions f and g in $\mathcal{F}(S,F)$ are equal $\Leftrightarrow f(s) = g(s)\ \forall s \in S$. So, we have:

$$f(0) = 2 \cdot 0 + 1 = 1 = 1 + 4 \cdot 0 - 2 \cdot 0^2 = g(0)$$
$$f(1) = 2 \cdot 1 + 1 = 3 = 1 + 2 = 1 + 4 \cdot 1 - 2 \cdot 1^2 = g(1)$$

Therefore we have $\forall s \in S = \{0,1\},\ f(s) = g(s) \Leftrightarrow f = g$
Likewise, we have:

$$h(0) = 5^0 + 1 = 1 + 1 = 2 = 1 + 1 = g(0) + f(0)$$
$$h(1) = 5^1 + 1 = 5 + 1 = 6 = 3 + 3 = f(1) + g(1)$$

Therefore we have $\forall s \in S = \{0,1\},\ h(s) = f(s) + g(s) \Leftrightarrow h = f + g$.

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Exercise 18 page 15: *Let* $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. *For* $(a_1, a_2), (b_1, b_2) \in V$ *and* $c \in \mathbb{R}$, *we define:*

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$
$$c \cdot (a_1, a_2) = (c \cdot a_1, c \cdot a_2)$$

Is V *a vector space over* $\mathbb{R}$ *with these operations? Justify your answer.*

V is a not a vector space over $\mathbb{R}$ with these operations since it does not satisfy the commutative property of addition. In fact, for $a_1, a_2, b_1, b_2 \in \mathbb{R}$, we generally have:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \neq (b_1 + 2a_1, b_2 + 3a_2) = (b_1, b_2) + (a_1, a_2)$$

For example, $0,1 \in \mathbb{R}$ so $(1,0) \in V \wedge (0,1) \in V$ but we have

$$(0,1) + (1,0) = (0 + 2 \cdot 1, 1 + 3 \cdot 0) = (2,1) \neq (1,3) = (1 + 0 \cdot 2, 0 + 3 \cdot 1) = (1,0) + (0,1)$$

Therefore, $\forall x, y \in V$, we do not have always $x + y \in V$

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Exercise 21 page 16: *Let V, W be vector spaces over a field F. Let*

$$Z = \{(v, w) : v \in V \wedge w \in W\}$$

For $v_1, v_2 \in V, w_1, w_2 \in W$ *and* $c \in F$, *we define:*

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$c \cdot (v_1, w_1) = (c \cdot v_1, c \cdot w_1)$$

Prove that Z is a vector space over F.

- **Closure**

First we have to show that $Z = V \times W$ is closed under addition and scalar multiplication.

For addition, let $v_1, v_2 \in V, w_1, w_2 \in W$, then:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

Moreover, $v_1, v_2 \in V \wedge w_1, w_2 \in W \Rightarrow v_1 + v_2 \in V \wedge w_1 + w_2 \in W$ since V and W are vector spaces. So $(v_1 + v_2, w_1 + w_2) \in Z$ and is unique. Hence we have that $+ : Z \times Z \to Z$.

For scalar multiplication, let $v_1 \in V, w_1 \in W$ and $c \in F$:

$$c \cdot (v_1, w_1) = (c \cdot v_1, c \cdot w_1)$$

So, $v_1, \in V, w_2 \in W, c \in F \Longrightarrow c \cdot v_1 \in V \wedge c \cdot w_2 \in W$ since V, W are vector spaces over F. Thus $(c \cdot v_1, c \cdot w_1) \in Z$ and is unique. Hence: $\cdot: F \times Z \to Z$.

- **VS 1: commutative property of addition**

For any $v_1, v_2 \in V, w_1, w_2 \in W,$ we have

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1)$$

because V, W are vector spaces and addition is commutative in a vector space. So VS1 holds for Z since we have $\forall\ v_1, v_2 \in V, \forall w_1, w_2 \in W$:

$$(v_1, w_1) + (v_2, w_2) = (v_2, w_2) + (v_1, w_1)$$

- **VS 2: associativity of addition:**

For any $v_1, v_2, v_3 \in V, w_1, w_2, w_3 \in W,$ we have :

$$\big((v_1, w_1) + (v_2, w_2)\big) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = \big((v_1 + v_2) + v_3, (w_1 + w_2) + w_3\big)$$

Moreover, since V, W are vector spaces we have $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ and $(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$ by property VS2 of vector spaces. So:

$$\big((v_1 + v_2) + v_3, (w_1 + w_2) + w_3\big) = \big(v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)\big)$$
$$= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + \big((v_2, w_2) + (v_3, w_3)\big)$$

Finally, we have: $\big((v_1, w_1) + (v_2, w_2)\big) + (v_3, w_3) = (v_1, w_1) + \big((v_2, w_2) + (v_3, w_3)\big)$ for any $v_1, v_2, v_3 \in V, w_1, w_2, w_3 \in W$. Thus addition on Z is associative.

- **VS 3: existence of identity element for addition:**

Let $v_1, \in V, w_2 \in W$. Since V is a vector space $\exists 0_V \in V$ such that $v_1 + 0_V = v_1$. Analogously, since W is a vector space $\exists 0_W \in W$ such that $w_1 + 0_W = w_1$.
Moreover since, $0_V \in V\ 0_W \in W, 0_Z := (0_V, 0_W) \in Z$. So for any $v_1, \in V, w_2 \in W$ we have:

$$(v_1, w_1) + 0_Z = (v_1 + 0_V, w_1 + 0_W) = (v_1, w_1)$$

- **VS 4: existence of inverses for addition:**

Let $v_1, \in V, w_2 \in W$. Since V is a vector space $\exists v_2 \in V$ such that $v_1 + v_2 = 0_V$. Analogously, since W is a vector space $\exists w_2 \in W$ such that $w_1 + w_2 = 0_W$.
Moreover since, $v_2 \in V\ w_2 \in W$, $(v_2, w_2) \in Z$. Thus we have:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (0_V, 0_W) = 0_Z$$

Therefore, for any $(v_1, w_1) \in Z, \exists (v_2, w_2) \in Z$ such that $(v_1, w_1) + (v_2, w_2) = 0_Z$

- **VS 5: identity element for scalar multiplication:**

Let $v_1, \in V, w_2 \in W$. We have for $1 \in F$:

$$1 \cdot (v_1, w_1) = (1 \cdot v_1, 1 \cdot w_1) = (v_1, w_1)$$

Since V,W are vector spaces over $F \Longrightarrow 1 \cdot v_1 = v_1 \wedge\ 1 \cdot w_1 = w_1$

- **VS 6: associativity of scalar multiplication**

Let $v_1, \in V, w_2 \in W$ and $a, b \in F$, then we have:

$$a \cdot \big(b \cdot (v_1, w_1)\big) = a \cdot (b \cdot v_1, b \cdot w_1) = \big(a \cdot (b \cdot v_1), a \cdot (b \cdot w_1)\big)$$

And V,W vector spaces over $F \Longrightarrow a \cdot (b \cdot v_1) = (a \cdot b) \cdot v_1 \wedge\ a \cdot (b \cdot w_1) = (a \cdot b) \cdot w_1$.
So we have :

$$\big(a \cdot (b \cdot v_1), a \cdot (b \cdot w_1)\big) = \big((a \cdot b) \cdot v_1, (a \cdot b) \cdot w_1\big) = (a \cdot b) \cdot (v_1, w_1)$$

So for any $a, b \in F$ and $(v_1, w_1) \in Z,\ a \cdot \big(b \cdot (v_1, w_1)\big) = (a \cdot b) \cdot (v_1, w_1)$. Therefore the scalar multiplication is associative on Z.

- **VS 7: distributivity of scalar multiplication over addition of vectors**

For any $v_1, v_2 \in V, w_1, w_2 \in W$ and $a \in F$ we have:

$$a \cdot \big((v_1, w_1) + (v_2, w_2)\big) = a \cdot \big((v_1 + v_2, w_1 + w_2)\big) = \big(a \cdot (v_1 + v_2), a \cdot (w_1 + w_2)\big)$$

And V,W vector spaces over $F \Longrightarrow\ a \cdot (v_1 + v_2) = av_1 + av_2 \wedge a \cdot (w_1 + w_2) = aw_1 + aw_2$.
Therefore:

$$\big(a \cdot (v_1 + v_2), a \cdot (w_1 + w_2)\big) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2)$$
$$= a \cdot (v_1, w_1) + a \cdot (v_2, w_2)$$

For any $a \in F$ and $(v_1, w_1), (v_2, w_2) \in Z$ we have:

$$a \cdot \big((v_1, w_1) + (v_2, w_2)\big) = a \cdot (v_1, w_1) + a \cdot (v_2, w_2)$$

- **VS 8: distributivity**

For any $v_1 \in V, w_1 \in W$ and $a, b \in F$ we have:

$$(a + b) \cdot (v_1, w_1) = \big((a + b) \cdot v_1, (a + b) \cdot w_1\big)$$

And V, W vector spaces over $F \Longrightarrow\ (a + c) \cdot v_1 = av_1 + bv_1 \wedge (a + b) \cdot w_1 = aw_1 + bw_1$.
Therefore:

$$\big((a + b) \cdot v_1, (a + b) \cdot w_1\big) = (av_1 + bv_1, aw_1 + bw_1) = (av_1, aw_1) + (bv_1, bw_1)$$
$$= a \cdot (v_1, w_1) + b \cdot (v_1, w_1)$$

Hence, for any $a, b \in F$ and $(v_1, w_1) \in Z$ we have: $(a + b) \cdot (v_1, w_1) = a \cdot (v_1, w_1) + b \cdot (v_1, w_1)$.

To conclude Z satisfies all the conditions to be a vector space so Z is a vector space. Therefore, if $V, W$are vector spaces over a field F, then their Cartesian product $Z = V \times W$ is a vector space over F.

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Exercise 8 page 20: *Determine whether the following sets are subspaces of* $\mathbb{R}^3$ *under the operations of addition and scalar multiplication defined on* $\mathbb{R}^3$*. Justify your answer.*

Since the properties (VS 1), (VS2), (VS5), (VS6), (VS7), (VS8) holds for any vector in the vector space, they automatically hold when they are restricted to the vectors of the subsets.
Moreover, by theorem 1.3., it suffices to check:

(1) Closure under addition
(2) Closure under scalar multiplication
(3) 0 is in W_i

(a) $\boldsymbol{W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = 3a_2 \text{ and } a_3 = -a_2\}}$

$$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = 3a_2 \wedge\ a_3 = -a_2\} = \{(3a, a, -a) | a \in \mathbb{R}\}$$

(1) Since $0 \in \mathbb{R}$, we have: $0_{\mathbb{R}^3} = (0,0,0) = (3 \cdot 0,0,-0) \in W_1$.So W_1 is not empty.

(2) Let $a, b \in \mathbb{R}$ then we have:

$$(3a, a, -a) + (3b, b, -b) = a \cdot (3,1,-1) + b \cdot (3,1,-1) = (a + b) \cdot (3,1,-1)$$
$$= (3 \cdot (a + b), a + b, -a - b)$$

because the distributive property holds for any vector in $\mathbb{R}^3$.
Therefore, $(3a, a, -a) + (3b, b, -b) = (3 \cdot (a + b), a + b, -a - b) \in W_1$.
So $\forall x, y \in W_1, x + y \in W_1$. So it is closed under addition.

(3) Let $a, b \in \mathbb{R}$ then we have:

$$b \cdot (3a, a, -a) = (3(ba), ba, -ba)$$

Thus, $b \cdot x \in W_1 \ \ \forall\, x \in W_1, \forall b \in \mathbb{R}$.
Therefore, W_1 is closed under scalar multiplication.

Since W_1 is a subset of $\mathbb{R}^3$ and satisfies all the conditions of a vector space, W_1 is a subspace of $\mathbb{R}^3$.

(b) $\boldsymbol{W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = a_3 + 2\}}$

$$W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = a_3 + 2\} = \{(b + 2, a, b) | a, b \in \mathbb{R}^2\}$$

First notice that W_2 is not empty since $(2,0,0) \in W_2$.

Suppose $(b + 2, a, b), (d + 2, c, d) \in W_2$. Then, we have:

$$(b + 2, a, b) + (d + 2, c, d) = \big((d + b) + 4, a + c, b + d\big)$$

But, $\big((d + b) + 4, a + c, b + d\big) \notin W_2$ since $(d + b) + 4$ is not of the form $a_1 = a_3 + 2$.
Therefore W_2 is not closed under addition and W_2 is not a subspace of $\mathbb{R}^3$ by theorem 1.3.

(c) $\boldsymbol{W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | 2a_1 - 7a_2 + a_3 = 0\}}$

(1) We have $2 \cdot 0 - 7 \cdot 0 + 0 = 0$ so $(0,0,0) \in W_3$. So W_3 is not empty.

(2) Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ such that $2a_1 - 7a_2 + a_3 = 0 \wedge 2b_1 - 7b_2 + b_3 = 0$.
Then we have:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

We have $2(a_1 + b_1) - 7(a_2 + b_2) + (a_3 + b_3) = 2a_1 - 7a_2 + a_3 + 2b_1 - 7b_2 + b_3 = 0 + 0 = 0$.
Thus, $\forall x, y \in W_3, x + y \in W_3$. So W_3 is closed under addition.

(3) Let $(a_1, a_2, a_3) \in W_3$ and $c \in \mathbb{R}$. Then we have:

$$c \cdot (a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$$

Moreover we have $2ca_1 - 7ca_2 + ca_3 = c \cdot (2a_1 - 7a_2 + a_3) = c \cdot 0 = 0$.
Thus, $c \cdot x \in W_3 \ \ \forall\, x \in W_3, \forall c \in \mathbb{R}$ and W_3 is closed under scalar multiplication

Since W_3 is a subset of $\mathbb{R}^3$ and satisfies all the conditions of a vector space, W_3 is a subspace of $\mathbb{R}^3$.

(d) $\boldsymbol{W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 - 4a_2 - a_3 = 0\}}$

(1) We have $0 - 4 \cdot 0 - 0 = 0$ so $(0,0,0) \in W_4$. So W_4 is not empty.

(2) Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_4$. Then we have:
$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
Then we have:
$$a_1 + b_1 - 4(a_2 + b_2) - (a_3 + b_3) = (a_1 - 4a_2 - a_3) + (b_1 - 4b_2 - b_3) = 0$$

Thus, $\forall x, y \in W_4, x + y \in W_4$. So W_4 is closed under addition.

(3) Let $(a_1, a_2, a_3) \in W_4$ and $c \in \mathbb{R}$. Then we have:
$$c \cdot (a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$$

Moreover we have $ca_1 - 4ca_2 - ca_3 = c \cdot (a_1 - 4a_2 - a_3) = c \cdot 0 = 0$.
Thus, $c \cdot x \in W_4 \quad \forall\, x \in W_4, \forall c \in \mathbb{R}$ and W_3 is closed under scalar multiplication

Since W_4 is a subset of $\mathbb{R}^3$ and satisfies all the conditions of a vector space, W_4 is a subspace.

(e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 - 2a_2 - 3a_3 = 1\}$

Here we have that $0 - 2 \cdot 0 - 3 \cdot 0 = 0 \neq 1$. Therefore $0_{\mathbb{R}^3} \notin W_5$. It implies that W_5 is not a subspace of $\mathbb{R}^3$ since by theorem 1.3. the zero vector of W_5 must be the same as the zero vector of $\mathbb{R}^3$ if W_5 is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

In fact, since we have $\forall x \in \mathbb{R}^3$, $0_{\mathbb{R}^3} + x = x$. So it is in particular true for all elements of W_5 since it is a subset of $\mathbb{R}^3$.
Supposing that W_5 is a subspace, it has a zero vector such that $\forall x \in W_5$, $0_{W_5} + x = x$.
Supposing W_5 is a subspace, we get by cancellation laws $\forall x \in W_5$, $0_{W_5} + x = x = 0_{\mathbb{R}^3} + x \implies 0_{W_5} = 0_{\mathbb{R}^3}$. Moreover, in a vector space the zero vector is unique.
Therefore, if $0_{\mathbb{R}^3}$ is not in W_5, W_5 does not satisfies the property $VS3$ of vector spaces and thus it is not a subspace of $\mathbb{R}^3$.

(f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | 5a_1^2 - 3a_2^2 - 6a_3^2 = 0\}$

First we notice that $(0,0,0)$ is in W_6 because $5 \cdot 0^2 - 3 \cdot 0^2 - 6 \cdot 0^2 = 0$, so W_6 is not empty.

Suppose $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_6$. Then we have:
$$x = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
But we have:
$$\begin{aligned} &5(a_1 + b_1)^2 - 3(a_2 + b_2)^2 - 6(a_3 + b_3)^3 \\ &= 5(a_1^2 + 2a_1b_1 + b_1^2) - 3(a_2^2 + 2a_2b_2 + b_2^2) - 6(a_3^2 + 2a_3b_3 + b_3^2) \\ &= (5a_1^2 - 3a_2^2 - 6a_3^2) + (5b_1^2 - 3b_2^2 - 6b_3^2) + 10a_1b_1 - 6a_2b_2 - 12a_3b_3 \\ &= 0 + 0 + 10a_1b_1 - 6a_2b_2 - 12a_3b_3 = 10a_1b_1 - 6a_2b_2 - 12a_3b_3 \neq 0 \end{aligned}$$

Therefore W_6 is not closed under addition, which implies that W_6 is not a subspace of $\mathbb{R}^3$ by theorem 1.3.

Exercise 19 page 21: *Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2 \vee W_2 \subseteq W_1$.*

Let W_1 and W_2 be subspaces of a vector space V over F.

Let us show the left implication $(\Leftarrow)$.

- Suppose that $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$. Moreover, since W_2 is a subspace of V then $W_1 \cup W_2$ is a subspace of V.

- Suppose that $W_2 \subseteq W_1$. Then $W_1 \cup W_2 = W_1$. Since W_1 is a subspace of V then $W_1 \cup W_2$ is a subspace of V.

Hence $W_1 \subseteq W_2 \vee W_2 \subseteq W_1 \Longrightarrow W_1 \cup W_2$ is a subspace of V.

Let us show the right implication $(\Longrightarrow)$.

Suppose that $W_1 \cup W_2$ is a subspace of a vector space V over F with W_1, W_2 subspaces of a vector space V over F.

Suppose that W_1 is not included in W_2 and that W_2 is not included in W_1. We have that:

- W_1 is not included in $W_2 \Longrightarrow W_1 \setminus W_2 \neq \emptyset$
- W_2 is not included in $W_2 \Longrightarrow W_2 \setminus W_1 \neq \emptyset$

Thus, let $x \in W_1 \setminus W_2$ and $y \in W_2 \setminus W_1 \Longrightarrow x, y \in W_1 \cup W_2 \Longrightarrow x + y \in W_1 \cup W_2$ since $W_1 \cup W_2$ is a subspace.

Moreover $x + y \in W_1 \cup W_2 \Longrightarrow x + y \in W_1 \; or \; x + y \in W_2$.

- Suppose $x + y \in W_1$. Since $x \in W_1 \setminus W_2$ and that W_1 is a vector space $-x \in W_1$. Moreover a vector space is closed under addition so $x + y - x \in W_1$ and thus $y \in W_1$. This contradicts the fact that $y \in W_2 \setminus W_1$.
- Suppose $x + y \in W_2$. Since $y \in W_2 \setminus W_1$ and that W_2 is a vector space $-y \in W_2$. Moreover a vector space is closed under addition so $x + y - y \in W_2$ and thus $x \in W_1$. This contradicts the fact that $x \in W_1 \setminus W_2$.

So for W_1, W_2 subspaces of a vector space V, $W_1 \cup W_2$ is a subspace of $V \Longrightarrow (W_2 \subseteq W_1) \vee (W_1 \subseteq W_2)$.

To conclude we have that if W_1 and W_2 are subspaces of a vector space V over F:

- $W_1 \cup W_2$ is subspace of a vector space V over $F \Longrightarrow W_2 \subseteq W_1 \vee W_1 \subseteq W_2$
- $W_2 \subseteq W_1 \vee W_1 \subseteq W_2 \Longrightarrow W_1 \cup W_2$ is subspace of a vector space V over F

Hence if W_1 and W_2 are subspaces of a vector space V over F, $W_1 \cup W_2$ is subspace of a vector space V over $F \Longleftrightarrow W_2 \subseteq W_1 \vee W_1 \subseteq W_2$.

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