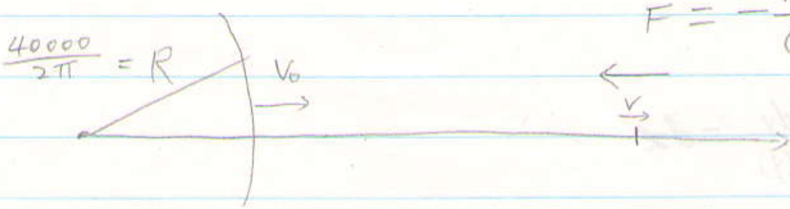
$m(x)\,dx + n(y)\,dy = 0 \Rightarrow M(x) + N(y) = C$

"separated" $\longleftrightarrow m + ny' = 0$

$\frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

example 2 (escape velocities)

$F = -\frac{mgR^2}{(R+x)^2} \Rightarrow \frac{dv}{dt} = -\frac{gR^2}{(R+x)^2}$



$\Rightarrow v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{R+x} + C$

$V(0) = V_0$

At $x=0$ $\frac{V_0^2}{2} = gR + C \Rightarrow C = \frac{V_0^2}{2} - gR$

$\frac{V^2}{2} = \frac{gR^2}{R+x} + \frac{V_0^2}{2} - gR$

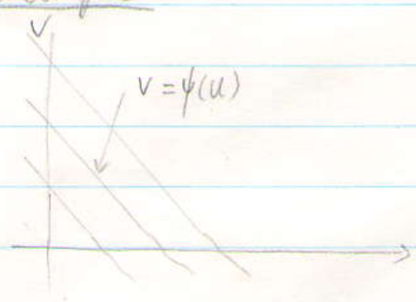
For which V_0, $\lim_{x\to\infty} v(x) = 0$?

$0 = 0 + \frac{V_0^2}{2} - gR$

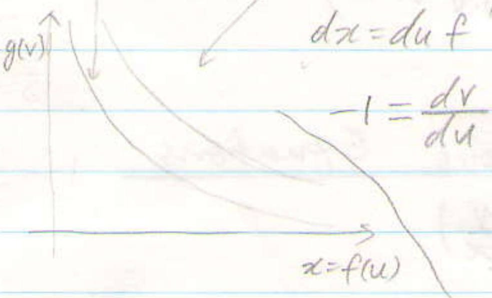
$\Rightarrow V_0 = \sqrt{2gR}$

$\frac{dx}{du} = f'$, $\frac{dy}{dv} = g'$

example



$x = e^u$
$y = e^v$



$dx = du\,f'$, $dy = dv\,g'$

$-1 = \frac{dv}{du} = \frac{dy}{g'} \Big/ \frac{dx}{f'}$

$= \frac{dy}{dx}\cdot\frac{f'}{g'}$

$u + v = c$

$\frac{dv}{du} = -1$

$\frac{dy}{dx} = -\frac{g'}{f'} \Rightarrow \frac{dy}{g'} + \frac{dx}{f'} = 0$

$v = \psi(u)$

$g^{-1}(y) = \psi(f^{-1}(x))$

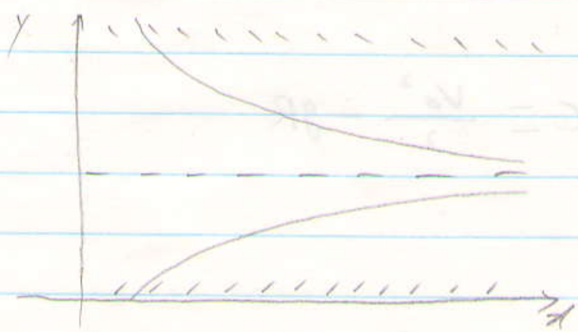
$y = g(\psi(f^{-1}(x)))$

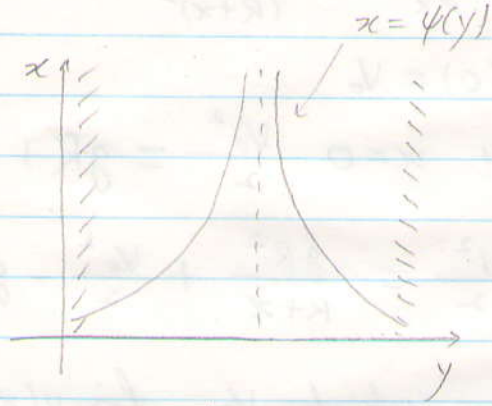
$\phi = g \circ \psi \circ f^{-1}$

Autonomous Equations

$y' = f(y)$. $\frac{dy}{dx} = f(y)$. $\frac{dy}{f(y)} = dx$

$\frac{y'}{f} - 1 = 0$





$\psi' = g(y)$

$\frac{dy}{dx} = f(y)$ $\frac{dx}{dy} = \frac{1}{f(y)}$

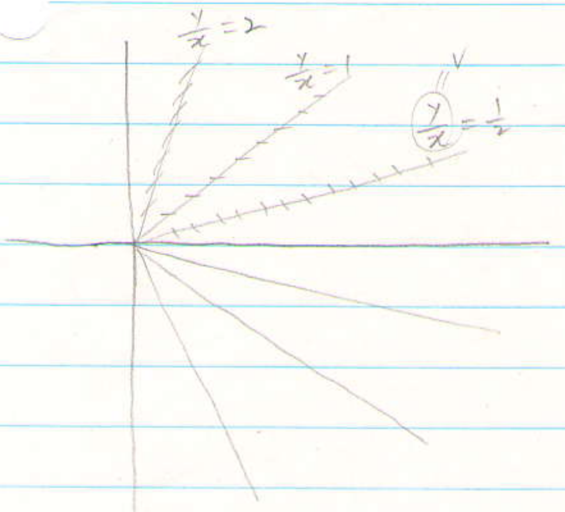
$x' = \frac{1}{f(y)} = g(y)$

$x = \int g(y)\,dy$

Homogeneous Equations

$y' = F(\frac{y}{x})$

$y' = \frac{y^2 + 2xy}{x^2} = \frac{y^2/x^2 + 2\frac{y}{x}}{1} = (\frac{y}{x})^2 + 2\frac{y}{x}$



<u>Solution</u>

$v = \frac{y}{x} \Rightarrow y = v \cdot x$

$y' = v'x + v$

$y' = F\left(\frac{y}{x}\right) \Rightarrow v' \cdot x + v = F(v)$

$xv' = F(v) - v$

$x\frac{dv}{dx} = F(v) - v$

$\frac{dv}{F(v) - v} = \frac{dx}{x}$

$y' = \left(\frac{y}{x}\right)^2 + 2 \cdot \frac{y}{x}$. $y = x \cdot v$

$xv' + v = v^2 + 2v$

$xv' = v^2 + v = v(v+1)$

$\frac{dv}{v(v+1)} = \frac{dx}{x}$

$A(v+1) + Bv = 1 \Rightarrow A = 1, B = -1$

$\log|x| + C = \int \frac{dv}{v(v+1)} = \int \left(\frac{A}{v} + \frac{B}{v+1}\right) dv$

$\log|x| + C = \int \left(\frac{1}{v} - \frac{1}{v+1}\right) dv$

$$\Rightarrow \log|x| + C = \log|v| - \log|v+1|$$

$$= \log\left|\frac{v}{v+1}\right|$$

$$\Rightarrow C \cdot x = \frac{v}{v+1} \Rightarrow v = \frac{-Cx}{Cx-1}$$

$$\Rightarrow y = \frac{-Cx^2}{Cx-1}$$