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# Determinants

det : $M_{n\times n}(F) \longrightarrow F$  $A \in M_{n\times n}(F)$

$\det(A) = |A| \in F$

1. Usefulness
2. Formula
3. "Axiomatic" properties
   a
   b
   c

① THM: $A$ is invertible iff $\det A \neq 0$ (det is invertible)

② det, $|\ |$ is defined recursively for $n\times n$ matrices in terms of $(n-1)\times(n-1)$ matrices

I $\det(a_{11}) = a_{11}$  $\det(7) = 7$

II If $A = (A_{ij})$ is $n\times n$, define

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \det(\widetilde{A}_{ij})$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$\widetilde{A}_{ij}$ := the matrix you get from $A$ by removing $i$th row & $j$th column

$= 1(5\cdot 9 - 4\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7)$

$-3 - 2(-6) + 3(-3) = -3 + 12 - 9 = 0 \implies A$ is not invertible

Example:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det(d) - b \det(c)$$

$$= a \cdot d - b \cdot c$$

checking: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{1}{a_{11}a_{22}a_{33}} + \underset{2}{a_{12}a_{23}a_{31}} + \underset{3}{a_{13}a_{21}a_{32}}$

$$- a_{13}a_{22}a_{33} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

Properties:

(a) $\det(I) = 1$

(b) det is multi-linear in the rows (rows can be thought of vectors) so $r_j = a r_j' + b r_j''$

$$\det\begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ r_j \\ \vdots \\ — r_n — \end{pmatrix} = a \det\begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_j' — \\ \vdots \\ — r_n \end{pmatrix} + b \det\begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_j'' — \\ \vdots \\ — r_n \end{pmatrix}$$

~~det (A+B) = det A + det B~~ not true

an example to show multi linearity (nothing to do with determinants)

$x \cdot y$ is "multilinear" in $x$ & $y$

$(5+3)y = 5y + 3y$ but $(5+3)(7+9) \neq 5 \cdot 7 + 3 \cdot 9$

$x(7+9) = x \cdot 7 + x \cdot 9$

ⓒ If ~~#~~ a pair of adjecent rows of $A$ are equal, $\det A = 0$. $\begin{vmatrix} - r_1 - \\ - r_1 - \\ \vdots \end{vmatrix} = 0$

<u>Theorem:</u> It is easy to tell how dets change under row operations

**1** Interchaning two rows

$$\det(E^1_{ij} A) = -\det(A) \Rightarrow \det(E^1_{i,j}) = -1$$

**2.** Multiplying a row by $C$

en for
-0

$$\det(E^2_{i,c} \cdot A) = c \cdot \det A \Rightarrow \det E^2_{i,c} = C$$

3. $\det(E^3_{i,j,c} A) = \det(A) \Rightarrow \det(E^3_{i,j,c}) = 1$

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= -(-2) \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2.1 = 2$$

Moral: Using elementary row ops we can reduce the computation of any det to the computation of the det of a matrix in r.r.e.f.

$$\det A' = \begin{cases} 0 & A' \text{ has a row of } 0's \\ & \text{otherwise} \\ 1 & A = I \end{cases}$$

Moral: using prop., we can compute any det without going back to formulas.

Moral: $\det A \neq 0$ iff $A$ is invertible. (most important prop.)

prf: Do row reduction without ever multiplying a row by 0. Find

$\det A = \underbrace{\gamma}_{\neq 0} \cdot \det(\overrightarrow{EA})$ is in r.r.e.f.

$\gamma$ is a product of c's & (-1)'s so $\neq 0$.

$EA =$ $\det(EA) \neq 0 \Leftrightarrow \det A \neq 0$

$\Updownarrow$

A is invertible

proof of thm from "Axiomatic" properties

(2) follow from b taking $r'' = 0$ or $b = 0$

$$\left|\begin{pmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_j + 7_{j+1} - \\ - r_{j+1} - \\ \vdots \\ - r_n - \end{pmatrix}\right| \underset{(b)}{\overset{by}{=}} \left|\begin{pmatrix} - r_1 - \\ \vdots \\ - r_j - \\ - r_{j+1} - \\ \vdots \\ - r_n - \end{pmatrix}\right| + 7 \left|\begin{pmatrix} - r_1 - \\ \vdots \\ - r_{j+1} - \\ \vdots \\ - r_n - \end{pmatrix}\right| \underset{(c)}{\overset{by}{=}} \left|\begin{pmatrix} - r_1 - \\ \vdots \\ - r_n - \end{pmatrix}\right|$$

$\Rightarrow$ (3) holds if the two rows are adjacent.

continued on

$$\begin{vmatrix} = \\ \vdots \\ -r'- \\ -r''- \\ \vdots \end{vmatrix} \underset{\text{prev}}{\overset{\text{by}}{=}} \begin{vmatrix} = \\ r' \\ r''+r' \\ = \end{vmatrix} \underset{\text{from } 1^{st}}{\overset{\text{subtract } 2^{nd}}{=}} \begin{vmatrix} = \\ \vdots \\ -r'' \\ r''+r' \\ = \end{vmatrix} \underset{\text{to } 2^{nd}}{\overset{\text{add } 1^{st}}{=}} \begin{vmatrix} = \\ -r'' \\ r' \\ \vdots \\ = \end{vmatrix}$$

$$\overset{\text{by}}{\underset{2}{=}} - \begin{vmatrix} = \\ r'' \\ r' \\ = \end{vmatrix} \Rightarrow \text{(1) holds for adjecent rows}$$

$\Rightarrow$ (3) holds even if the relevant rows ($r_i$ & $r_j$) are not adjecent.

$$\begin{vmatrix} - \\ r_i \\ = \\ r_j \\ = \end{vmatrix} 3m = \pm \begin{vmatrix} = \}m \\ r_i \\ r_j \\ = \end{vmatrix} = \pm \begin{vmatrix} = 3m \\ r_i \\ r_j + 7r_i \\ = \end{vmatrix} = \begin{vmatrix} - \\ r_i \\ = \; 3m \\ r_j + 7r_i \\ = \end{vmatrix}$$

$\Rightarrow$ (1) holds for any pair of rows

proof $\alpha$ use computer trick again

Proof $\beta$

$$\begin{vmatrix} - \\ r_i \\ = \}m \\ r_j \\ = \end{vmatrix} \underset{\text{switches}}{\overset{2m+1}{=}} (-1)^{2m+1} \begin{vmatrix} r_j \\ r_i \end{vmatrix} = - \begin{vmatrix} r_j \\ r_i \end{vmatrix}$$