

## MAT 257

Metric:  $d: X \times X \rightarrow \mathbb{R}$  st. i)  $d(x,y) = d(y,x)$ . ii)  $d(x,y) \geq 0 \text{ & } d(x,y) = 0 \iff x=y$ . iii)  $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$   
 $\Sigma\text{-nbd: } U(x_0, \varepsilon) = \{x \in X : d(x_0, x) < \varepsilon\} \quad U \subset X \text{ is open if } \forall x_0 \in U, \exists \varepsilon > 0 \text{ st. } U(x_0, \varepsilon) \subset U. \quad F \subset X \text{ closed if } X \setminus F \text{ open}$

\*  $U(x_0, r)$  is open. (take  $\Sigma = r - d(x_0, y)$  for each  $y \in U(x_0, r)$ )  $\emptyset, X$  are open & closed.

$\forall \alpha \in I, U_\alpha \text{ open} \Rightarrow \bigcup_{\alpha \in I} U_\alpha \text{ open}; \quad \forall \{i_1, \dots, i_n\}, U_i \text{ open} \Rightarrow \bigcap_{i=1}^n U_i \text{ is open.}$

$\forall \alpha \in I, F_\alpha \text{ closed} \Rightarrow \bigcup_{\alpha \in I} F_\alpha \text{ closed}, \quad \forall \{i_1, \dots, i_n\}, F_i \text{ closed} \Rightarrow \bigcap_{i=1}^n F_i \text{ closed}$

\* In  $\mathbb{R}^m$ ,  $d_1(x,y) = \|x-y\|$ ,  $d_2(x,y) = |x-y|$  then  $U$  is  $d_1$  open  $\Leftrightarrow U$  is  $d_2$  open;  $F$  is  $d_1$  closed  $\Leftrightarrow F$  is  $d_2$  closed  
 $X, Y$  metric space,  $T \subset X$ ;  $U \subset T$  open  $\Leftrightarrow \exists V \subset X, V$  open in  $X$  and  $U = V \cap T$

$F \subset T$  closed  $\Leftrightarrow \exists G \subset X, G$  closed in  $X$  and  $F = G \cap T$

Limit point:  $x_0 \in X$  is a lp of  $A \subset X$  if  $\forall \varepsilon > 0$   $(U(x_0, \varepsilon) \cap A) \setminus \{x_0\} \neq \emptyset \Leftrightarrow \forall \varepsilon > 0, |U(x_0, \varepsilon) \cap A| = \infty$

Closure:  $\bar{A} := A \cup \{\text{limit pts of } A\} = A \cup p(A)$

\*  $A$  is closed  $\Leftrightarrow \bar{A} = A$  (show  $A^c$  open  $\Leftrightarrow \bar{A} = A$ )  $\bar{A}$  is smallest closed set containing  $\bar{A}$  &  $U$  closed sets containing  $A$

Continuous:  $f: X \rightarrow Y, x_0 \in X$ .  $f$  cont. at  $x_0$  if  $\exists$  nbd  $U$  of  $x_0$  st.  $f(U) \subset V$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$  st.  $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

\* For  $f: X \rightarrow Y$ ,  $F$  is cont.  $\Leftrightarrow \forall$  open  $V \subset Y, f^{-1}(V)$  is open  $\Leftrightarrow \forall$  closed  $F$  in  $Y, f^{-1}(F)$  is closed.

$\text{Int } A = \{x \in A : \exists \varepsilon > 0, U(x, \varepsilon) \subset A\} = \text{Union of all open sets contained in } A = \text{maximal open set contained in } A$

$\text{Ext}(A) = \text{int}(A^c) \quad \text{Bd}(A) = X \setminus (\text{int } A \cup \text{ext } A)$

\*  $\text{ext}(A) = X \setminus \bar{A} \quad \text{int } A = X \setminus \bar{A}^c \quad \text{Bd } A = \bar{A} \cap \bar{A}^c$

Compact:  $X$  is compact if  $(X = \bigcup_{\alpha \in I} U_\alpha, U_\alpha \text{ open} \Rightarrow \exists \text{ finite } n \text{ st. } X = \bigcup_{i=1}^n U_{\alpha_i} \text{ where } \alpha_i \in I)$

$A \subset X$  is compact if  $(A \subset \bigcup_{\alpha \in I} U_\alpha, U_\alpha \text{ open} \Rightarrow \exists \text{ finite } n \text{ st. } A \subset \bigcup_{i=1}^n U_{\alpha_i} \text{ where } \alpha_i \in I)$

\* cont. function on a compact space is bdd.  $[0,1]$  is compact.  $(0,1]$  is not.

Subset  $A \subset \mathbb{R}^m$  is compact  $\Leftrightarrow A$  is closed & bdd.  $X \times Y$  compact  $\Leftrightarrow X \times Y$  is compact.

\*  $f: X \rightarrow Y$  cont. &  $X$  is compact, then  $f(X)$  is compact. A closed subset  $T$  of compact set  $A$  is compact

Connected:  $X$  is conn. if # clopen sets in  $X$  expect  $\emptyset$  and  $X$ .

\*  $A \subset \mathbb{R}$  is conn. iff  $A$  is a generalised interval.  $X, Y$  conn.  $\Rightarrow X \times Y$  is conn.

$X$  conn. &  $f: X \rightarrow Y$  cont.  $\Rightarrow f(X)$  is conn.

\*  $f: \mathbb{R}^m \rightarrow \mathbb{R}$  continuous function &  $\exists a, b \in \mathbb{R}^m$  st.  $f(a) < 0 < f(b)$ , then  $\exists x_0 \in \mathbb{R}^m$  st.  $f(x_0) = 0$

Directional Derivatives:  $f: \mathbb{R}^n \rightarrow \mathbb{R}, f'(a; u) = \lim_{h \rightarrow 0} \frac{f(a+uh) - f(a)}{h} \quad a, u \in \mathbb{R}^n$

Differential:  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is diff at  $a \in \mathbb{R}^m$  if  $\exists B \in M_{m \times n}(\mathbb{R})$  st.  $\|f(a+h) - f(a) - Bh\| / \|h\| \xrightarrow{h \rightarrow 0} 0$

Another way:  $f(a+h) - f(a) - Bh \in O(h)$  where  $O(h) = \{\phi: \mathbb{R}^m \rightarrow \mathbb{R}^n; \|\phi(h)\| / \|h\| \rightarrow 0 \text{ as } h \rightarrow 0\}$

\*  $B$  exists  $\Rightarrow$  unique and  $Df(a) = B$ ;  $f$  is diff. then  $f'(a; u) = Df(a) \cdot u$  then  $Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$

Class:  $f$  is of class  $C^r$  on  $A \subset \mathbb{R}^m$  if  $\frac{\partial^r f}{\partial x_i^r}$  exists & are of class  $C^{r-1}$  (ren\*) ;  $f$  is  $C^0$  if  $f$  is continuous.

\* If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^2$ , then for any  $i, j$ ,  $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right) \quad \partial_i \partial_j f = \partial_j \partial_i f$

$$D(g \circ f)(a) = Dg(f(a)) \cdot Df(a)$$



\*MVT: A open in  $\mathbb{R}^m$ ,  $f: A \rightarrow \mathbb{R}$  diff. on A. If line  $a$  to  $a+h$  is contained in A, then  $\exists c = a + th, 0 < t < 1$  where  $f(a+h) - f(a) = Df(c) \cdot h$

P20 **Inverse FT**:  $A \subset \mathbb{R}^n$  open,  $f: A \rightarrow \mathbb{R}^m$  of class  $C^r$ . If  $Df(x)$  is non-singular at the point  $x$  of A, then there exists a nbd  $U$  of  $x$  s.t.  $f$  carries  $U$  in a 1-1 fashion onto an open set  $V$  of  $\mathbb{R}^m$  & the inverse function is  $C^r$ .

\*  $\forall \varepsilon > 0, \exists \text{ nbd } J = J_\varepsilon = U(0, \delta) \text{ s.t. } \forall x, y \in J, \|f(y) - f(x) - (y-x)\| \leq \varepsilon \|y-x\|$

P24 \* **Implicit FT**: Given a  $C^r$  function  $f: \mathbb{R}_{x_1, \dots, x_n} \times \mathbb{R}_{x_1, \dots, x_k} \rightarrow \mathbb{R}^k$  and  $(a, b) \in \mathbb{R}^n \times \mathbb{R}^k$  s.t.  $f(a, b) = 0$  and  $df/dy$  is non-singular, then  $\exists$  unique  $g: (\text{nbd}(a)) \rightarrow (\text{nbd}(b))$  s.t.  $g(a) = b$  &  $\forall x \in U, f(x, g(x)) = 0$ .  $g$  is  $C^r$ .

Refinement:  $P' = (a = t_0 < t_1 < \dots < t_n = b)$  is a ref. of  $P = (a = s_0 < s_1 < \dots < s_k = b)$  if  $s_j \in P' \forall 0 \leq j \leq k$

\*  $P'$  refines  $P \Rightarrow L(f, P') \geq L(f, P) \text{ & } U(f, P') \leq U(f, P) \Rightarrow L(f, P) \leq U(f, P')$

\* Every continuous function on Q is integrable.  $f$  integrable  $\Leftrightarrow \forall \varepsilon > 0, \exists P$  s.t.  $U(f, P) - L(f, P) < \varepsilon$

Cont:  $\forall x \in Q, \forall \varepsilon > 0, \exists \delta > 0$  s.t.  $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ .

Unif. cont:  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in Q, |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

\* Every unif. cont. function on Q is integrable; Every cont. function on a compact set is unif. cont.

\* Lebesgue number lemma:  $\{U_\alpha\}$  open cover of compact  $(X, d)$  then  $\exists \delta > 0$  s.t. any  $B = U(x, \delta)$  contained in some  $U_\alpha$

Measure-0:  $A \subset \mathbb{R}^n$  is measure-0 if  $\forall \varepsilon > 0 \exists$  countable collection R in  $\mathbb{R}^n$  s.t. i)  $A \subset \bigcup_i R_i$ ; ii)  $\sum_{i=1}^{\infty} V(R_i) < \varepsilon$

P29 \* A bdd function  $f: Q \rightarrow \mathbb{R}$  is integrable iff the discontinuous set of  $f$  is of measure-0.

$A$  is mea-0 &  $B \subset A \Rightarrow B$  is mea-0. A countable union of mea-0 is of mea-0

\*  $f: Q \rightarrow \mathbb{R}$  integrable, then  $f = 0$  almost always (except mea-0)  $\Rightarrow S_Q f = 0$ ;  $f \geq 0$  &  $\{x \in Q : f > 0\}$  not mea-0  $\Rightarrow S_Q f > 0$

\* Fundamental thm of calculus:  $f$  cont. on  $[a, b]$  then i)  $f(x) = \int_a^x f$ , then  $f'$  exists &  $f' = f$   
ii) If  $g = f$  then  $\int_a^b f = g(b) - g(a)$ .

\* Fubini:  $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$  rectangles,  $Q = A \times B, f: Q \rightarrow \mathbb{R}$  integrable then define  $U(x) = \int_{B \cap \{x\}} f$  &  $L(x) = \int_{A \cap \{x\}} f$   
we have  $S_A f = S_B f = S_{A \times B} f$

Indicator fcn of S: S bdd subset of  $\mathbb{R}^n$ ,  $S_S f = S_Q f 1_S, 1_S = \sum_{x \in S} \delta_x$ , Q is some rectangle  $\Delta S \subset Q$

\*  $f, g$  integrable over S,  $a, b \in \mathbb{R}$  then  $S_S af + bg = aS_S f + bS_S g$ ;  $f \leq g \Rightarrow S_S f \leq S_S g$ ;  $|S_S f| \leq S_S |f|$

$f \geq 0, T \subset S \Rightarrow S_T f \leq S_S f$ ;  $S_{S \setminus T} f = S_S f - S_T f$

Volume: S bdd & 1s integrable then  $V(S) = \int_S 1 = S_Q 1_S$ , say S is rectifiable; if  $f: S \rightarrow \mathbb{R}$  is bdd. cont. then  $\int_S f$  make sense

\*  $V(S) \geq 0$ ;  $S_1 \subset S_2 \Rightarrow V(S_1) \leq V(S_2)$ ;  $S_1, S_2$  recti  $\Rightarrow V(S_1 \cup S_2) = V(S_1) + V(S_2) - V(S_1 \cap S_2)$ ; S recti  $\& V(S) = 0 \Leftrightarrow S$  is mea-0

\* C compact recti. set in  $\mathbb{R}^n$  &  $f, g: C \rightarrow \mathbb{R}$  are cont. &  $f \leq g$  on C, then  $D = \{(x, t) : x \in C, f(x) \leq t \leq g(x)\} \subset \mathbb{R}^{n+1}$  recti,

$V(D) = \int_C g(x) - f(x)$ ; and if  $h: D \rightarrow \mathbb{R}$  is cont. then  $S_D h = \int_{C \times [0, 1]} h(x, t) dx dt$ .

$\int_A f = \sup \{ \int_A f : D \subset A \text{ compact & recti} \}$  for  $f \geq 0$ , otherwise  $\int_A f = \int_A f_+ - \int_A f_-$  where  $A \subset \mathbb{R}^n$  is open.

Converge to A: Say  $C_n \nearrow A$  if i)  $\forall n, C_n$  is compact & recti. ii).  $\forall n, C_n \subset \text{int}(C_{n+1})$  iii)  $\bigcup_{n=1}^{\infty} C_n = A$ .

\*  $C_n \nearrow A$ , then  $\int_A f$  exists iff  $\int_{C_n} f$  is a bdd seq. & then  $\int_A f = \lim_{n \rightarrow \infty} \int_{C_n} f$

\*  $A \subset \mathbb{R}^n$  bdd & open &  $f: A \rightarrow \mathbb{R}$  bdd & cont. then  $\int_A f$  exists. and if  $\int_A f$  exist we have  $\int_A f = \int_{\mathbb{R}^n} f$

\*  $A \subset \mathbb{R}^n$  open,  $f: A \rightarrow \mathbb{R}$  cont.  $U_1 \subset U_2 \subset \dots$  are open and  $\bigcup_{n=1}^{\infty} U_n = A$ , then  $\int_A f$  exists iff  $\int_{U_n} f$  bdd  
and then  $\int_A f = \lim_{n \rightarrow \infty} \int_{U_n} f$



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P38

\* Change of variable thm:  $g: A \rightarrow B$  diffeomorphism of open sets in  $\mathbb{R}^n$  (1-1, onto,  $C^1$  & inverse is  $C^1$ ) &  $f: B \rightarrow \mathbb{R}$  cont  
then  $S_{Bf}$  exists ( $\Rightarrow S_A(f \circ g) \cdot J_g$  exists and  $S_{Bf} = S_{A(f \circ g)} \cdot J_g$  where  $J_g: A \rightarrow \mathbb{R}$ ,  $J_g(x) = \det Dg(x)$ )

Parallelpiped spanned by  $v_1, \dots, v_n$ :  $v_1, \dots, v_n \in \mathbb{R}^n$ .  $P(v_1, \dots, v_n) = \{ \sum \alpha_i v_i \mid 0 \leq \alpha_i \leq 1 \ \forall i \}$ .

$$* \text{Vol}(P(v_1, \dots, v_n)) = |\det(v_1|v_2| \dots |v_n)|$$

Isometry:  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an isometry if  $\forall x, y \in \mathbb{R}^n$ ,  $d(hx), h(y)) = d(x, y)$

P41

\*  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an isometry iff it can be written as  $hx = P + Ax$  where  $P \in \mathbb{R}^n$ ,  $A \in M_{n \times n}$  s.t.  $A^T \cdot A = \text{Id}$

\* Isometries are: volume preserving;  $\{v_i\}$  (of  $A$ ) forms an orthonormal basis (as  $A^T \cdot A = \text{Id}$ )

Rotation matrices / orthogonal matrices  $O(n) = \{A \in M_{n \times n}(\mathbb{R}) \mid A^T \cdot A = \text{Id}\}$  form a group.

take  $h_i = h - h(0)$   
\*  $h_i$  preserves norms & inner products;  $A := (h(e_1) \mid h(e_2) \dots \mid h(e_n)) \in O(n)$ ;  $h_i$  is linear;  $h(\sum x_i e_i) = \sum x_i h(e_i)$

\* Gram-Schmidt process.  $\{u_i\}$  basis of an inner product space.  $\exists$  unique orthonormal basis  $\{v_i\}$  st.  $\forall k$ .

$$\left( \text{i.e. take } v'_1 = u_1, \dots, v'_k = u_k - \sum_{j=1}^{k-1} \langle u_k, v_j \rangle v_j; v_i = \frac{v'_i}{\|v'_i\|} \right) \quad \text{span}\{u_i\} = \text{span}\{v_i\}$$

P43

\*  $\exists$  unique  $V: (\mathbb{R}^n)^k \rightarrow \mathbb{R}_{\geq 0}$  st.  $(h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an orthogonal trans  $\& x_i \in \mathbb{R} \Rightarrow V(h(x_1) \dots h(x_k)) = V(x_1, \dots, x_k))$

$$\& (x_i \in \mathbb{R}^k \times \{0\}) \text{ i.e. } x_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix} \text{ where } y_i \in \mathbb{R}^k \Rightarrow V(x_1, \dots, x_k) = |\det(y_1| \dots |y_k)|$$

Further more:  $(V(x_1, \dots, x_k) = 0 \Leftrightarrow \{x_i\} \text{ dependent}) (X = (x_1 \mid \dots \mid x_k) \in M_{n \times k} \Rightarrow V(x_1, \dots, x_k) = |\det(X^T X)|^{\frac{1}{2}}$

TT 3. Manifold: parametrized  $k$ -manifold in  $\mathbb{R}^n$  is  $C^1$  map  $\alpha: A \rightarrow \mathbb{R}^n$ ,  $A$  open in  $\mathbb{R}^k$ ,  $\alpha(A) = Y$  the manifold.

$$V(Y) = V(Y, \alpha) := \int_A V(D\alpha) = \int_A \det((D\alpha)^T D\alpha)^{-\frac{1}{2}} \quad \int_Y f \, dv = \int_A (f \circ \alpha) V(D\alpha) = \int_B (f \circ \beta) V(D\beta)$$

\*  $\alpha: A \rightarrow \mathbb{R}^n$  manifold &  $g: B \rightarrow A$  diffeomorphism  $\Rightarrow \beta = \alpha \circ g$  manifold &  $\alpha(A) \cap \beta(B) \neq \emptyset$  &  $V(Y, \alpha) = V(Y, \beta)$

$k$ -dim manifold without boundary of  $C^r(r_2)$  in  $\mathbb{R}^n$ :  $M \subseteq \mathbb{R}^n$  st.  $\forall p \in M$ ,  $\exists$  open  $V$  in  $M$ ,  $p \in V$  & open  $U \subseteq \mathbb{R}^k$  and coordinate patch on  $M$  abt  $p$ :  $\alpha: U \rightarrow V$  (1-1, cont,  $\alpha^{-1}$  cont.)

a  $C^r$  homeomorphism  $\alpha: U \rightarrow V$  where  $D\alpha(x)$  rank  $k \ \forall x \in U$ .

$k$ -manifold (possibly with boundary):  $M \subseteq \mathbb{R}^n$  st.  $\forall x \in M \exists$  open  $U \subseteq \mathbb{R}^k$ :  $\{x \in U \mid x \in \partial M\}$  &  $C^r$  homeomorphism  $\alpha: U \rightarrow V$

$\partial M = \{p \in M \mid p = \alpha(q) \text{ for some } q \in \partial U\}$  where  $D\alpha$  of rank  $k \ \forall x \in U$ .

$f$  is  $C^r$  on  $S$ :  $(S \subseteq \mathbb{R}^k \& f: S \rightarrow \mathbb{R}^m)$ .  $\exists$  open  $U \subseteq \mathbb{R}^k$ ,  $S \subseteq U$  and a  $C^r$   $g: U \rightarrow \mathbb{R}^m$  st.  $g|_S = f$

\*  $f: S \rightarrow \mathbb{R}$  has property:  $\forall p \in S \exists U_p \ni p$  s.t.  $g: U_p \rightarrow \mathbb{R}$  st.  $f|_{S \cap U_p} = g|_{S \cap U_p} \Rightarrow f$  diffble on  $S$ .

\* Given  $A$  of opensets in  $\mathbb{R}^k$  whose overall union  $A = \bigcup_i U_i$ , then  $\exists \{\phi_i\}$  of non-neg compactly supp.  $C^\infty$  funcs s.t. 1.  $\forall i \exists U_i \in A$  st.  $\text{supp}(\phi_i) \subseteq U_i$  2.  $\phi_i(x) \geq 0$  for all  $x$ . 3.  $\sum \phi_i(x) = 1$

\*  $S \subseteq \mathbb{R}^k$ ,  $f: S \rightarrow \mathbb{R}^n$  is  $C^r$  ( $\forall p \in S, \exists$  nbd  $U_p \ni p$  s.t.  $g: U \rightarrow \mathbb{R}^n$  st.  $g|_{U \cap S} = f|_{U \cap S}$ )  $\Leftrightarrow \exists$  open  $U \subseteq \mathbb{R}^k$ ,  $S \subseteq U$  &  $g: U \rightarrow \mathbb{R}^n$  is  $C^r$  &  $g|_S = f$

\*  $\alpha: B^k \rightarrow B^n$  ( $k+n$ ) diff. able with  $\alpha^k$  diff. then  $(D\alpha)^T = D(\alpha^k)$

\*  $M^k$ :  $C^r$  mfld in  $\mathbb{R}^n$  &  $\alpha: U \rightarrow V$  is a coord patch, then  $\alpha^{-1}: V \rightarrow U$  is also  $C^r$

\* Given patch  $\alpha: U \xrightarrow{C^1} M \xrightarrow{C^1} \mathbb{R}^n$  and  $p = \alpha(a)$ ,  $\exists \pi: \mathbb{R}^n \rightarrow U$  such that  $g = \pi \circ \alpha$ ,  $g$  is a diff so in nbd. of  $q$

\* Transition function  $T = \alpha_2^{-1} \circ \alpha_1$  on  $\alpha_1^{-1}(V_1 \cap V_2)$  are  $C^r$  (where  $\alpha_i: U_i \xrightarrow{C^1} V_i \xrightarrow{C^1} \mathbb{R}^n$  coord. patch)

P48 \*  $A \subseteq \mathbb{R}^k$  open,  $f: A \rightarrow \mathbb{R}(C^r) \Rightarrow f^{-1}(h)$  is  $(m-d)$ -manifold for most  $h$  (if  $h \in \mathbb{R}$  st.  $Df(p) \neq 0 \dots$ )

$L^k(V) = \{f: V^k \rightarrow \mathbb{R} \mid \forall 1 \leq i \leq k, \forall v_1, \dots, v_k \in V, f(v_1, \dots, v_i, \dots, v_k) \text{ is linear}\}$

\*  $L^k(V)$  is a sub-vector space of  $\text{Fun}(V^k \rightarrow \mathbb{R})$  i.e.  $f, g \in L^k(V) \Rightarrow f+g \in L^k(V)$  &  $c f \in L^k(V)$



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P49 \*  $(a_1, \dots, a_n)$  basis of  $V$  &  $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k = n^k \Rightarrow \exists! \phi_I \in L^k(V)$  s.t.  $\phi_I(a_{j_1}, \dots, a_{j_k}) = \sum_{I=j}^1 \delta_{I,j}$  =  $\delta_{IJ}$   
 $\phi_I(v_1, \dots, v_k) = \phi_{i_1}(v_1) \cdots \phi_{i_k}(v_k)$   $\{\phi_I : I \in n^k\}$  is a basis of  $L^k(V)$  so  $\dim L^k(V) = n^k$

Tensor product  $\otimes : L^k(V) \times L^m(V) \rightarrow L^{k+m}(V)$  by  $(f \otimes g)(v_1, \dots, v_{k+m}) = f(v_1, \dots, v_k) \otimes g(v_{k+1}, \dots, v_m)$   
s.t.  $(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g$ ;  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ ;  $\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$

Dual transformation:  $T^* : L^k(W) \rightarrow L^k(V)$  by  $(T^* f)(v_1, \dots, v_k) = f(Tv_1, \dots, T v_k)$  where  $T : V \rightarrow W$  linear.

then: 1.  $T^*$  is linear. 2.  $T^*(f \otimes g) = T^*f \otimes T^*g$ . 3.  $S : W \rightarrow X$  linear then  $(S \circ T)^* f = T^*(S^* f)$ .

$\phi \in L^k(V)$  is alternating if  $\phi(x_1, \dots, x_k) = -\phi(x_k, \dots, x_1)$   $A^k(V) = \{\phi \in L^k(V) : \phi \text{ is alternating}\}$   
\* If  $\psi \in L^k(V)$  then  $\psi \in A^k(V) \Leftrightarrow \forall \sigma \in S_k \quad \psi^\sigma(v_1, \dots, v_k) := \psi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (-1)^{\sigma(i)} \psi(v_1, \dots, v_k)$  s.t.  $(-1)^{\sigma(i)} := 1, (-1)^{\sigma(i)} = (-1)^{\sigma(i)}$   
 $\eta_a^k = \{\eta_{(i_1, \dots, i_k)} \in \eta^k : i_1 < i_2 < \dots < i_k\}$   $|\eta_a^k| = \binom{n}{k}$   $(-1)^{\sigma(i)} := \prod_{j=1}^n \text{Sign}(i_j - \sigma(j))$

P52 \*  $\forall I \in \binom{n}{k} \exists! \psi_I \in A^k(V)$  s.t.  $\psi_I(a_J) = \delta_{IJ}$  &  $\{\psi_I\}_{I \in \binom{n}{k}}$  is a basis of  $A^k(V)$   
 $\psi_I = \sum_{\sigma \in S_k} (-1)^\sigma \psi_I^\sigma \in L^k(V)$ .

Wedge Product:  $f \in A^k, g \in A^l, (f \wedge g)(x_1, \dots, x_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (f \otimes g)^\sigma (-1)^\sigma = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f(x_{\sigma(1)}, \dots, x_{\sigma(k)}) (-1)^\sigma g(x_{\sigma(k+1)}, \dots, x_{\sigma(k+l)})$   
 $= \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(\dots) g(\dots) \quad \begin{matrix} \sigma_1 < \sigma_2 < \dots < \sigma_k \\ \sigma_{k+1} < \dots < \sigma_{k+l} \end{matrix}$

P54 \*  $\exists! \Lambda : A^k(V) \times A^l(V) \rightarrow A^{k+l}(V)$  s.t.  $(f \wedge g)\Lambda h = f \Lambda h + g \Lambda h$ ;  $(f \Lambda g) \Lambda h = f \Lambda (g \Lambda h) = (f \wedge g) \Lambda h$ ;  $f \wedge g = (-1)^{kl} g \wedge f$   
 $\psi_I = \phi_i, \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k}; T^*(f \wedge g) = (T^*f) \wedge (T^*g)$

Tangent vector  $\vec{z}$  to  $\mathbb{R}^n$  is  $\vec{z} = (x, v)$  where  $x, v \in \mathbb{R}^n$ . Tangent space  $T_x(\mathbb{R}^n) = \{(x, v) : v \in \mathbb{R}^n\}$   $\lambda(x, v) = (x, \lambda v)$

Directional derivative of  $f$  in  $\vec{z}$  is  $D_{\vec{z}}f = \lim_{h \rightarrow 0} \frac{f(x+hv) - f(x)}{h} = Df(x, v)$

$\alpha_* : T_x(\mathbb{R}^k) \rightarrow T_p(\mathbb{R}^k)$ ,  $\alpha_*(x, v) = (p, D_x(x) \cdot v)$  where  $A$  open in  $\mathbb{R}^k$  or  $H^k$ ,  $\alpha : A \rightarrow \mathbb{R}^k$  is  $C^r$  and  $x \in A$ ,  $p = \alpha(x)$

\*  $\alpha_*(x, v)$  is the velocity vector of curve  $y(t) = \alpha(x+tv)$  corresponding to  $t=0$ . (by chain rule)

P56 \*  $(\alpha \circ \beta)_* \vec{z} = \beta_* \circ \alpha_* \vec{z}$ ;  $D_{\vec{z}}(\alpha \circ \beta) = D\alpha_* \vec{z} (\beta)$ ;  $\alpha_*(y(t)) = (\alpha \circ y)(t)$ ;  $\alpha_* T_q_1 \mathbb{R}^k = \alpha_2 * T_{q_2} \mathbb{R}^k$

Tangent vector field (in  $A$ ):  $A$  open in  $\mathbb{R}^n$ ,  $C^r F : A \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  where  $F(x) = (x, f(x))$  where  $f : A \rightarrow \mathbb{R}^n$

Tangent space to  $M$  at  $p$ :  $T_p(M) = \alpha_*(T_x(\mathbb{R}^n))$  where  $M : C^r k\text{-mfld in } \mathbb{R}^n$ ;  $\alpha : U \rightarrow V$  corr. patch of  $p \in M$ ;  $x \in U$  s.t.  $x \in p$

$C^r$  vector field  $\gamma$ : 1.  $\forall i, y_i \in Y$  is  $C^r$  2.  $\forall f \in C^r$ ,  $\gamma f \in C^r$   $(\gamma f)(x) = D_x f = \sum y_i(x) \frac{\partial f}{\partial x_i}$

$k$ -tensor field on open  $A \subset \mathbb{R}^n$  is:  $w : A \rightarrow \bigcup_{x \in A} L^k(T_x \mathbb{R}^n)$  s.t.  $w(x) \in L^k(T_x \mathbb{R}^n)$   $w(x) = \sum_{I \in \binom{n}{k}} a_I(x) \phi_I$

value on a given  $k$ -tuple is  $w(x)(x_1, v_1), \dots, (x_k, v_k)$ ;  $\gamma_1, \dots, \gamma_k$  vector field then  $w(x)(\gamma_1, \dots, \gamma_k) = w(x)(\gamma_1(x), \dots, \gamma_k(x))$

$k$ -form on  $A \subset \mathbb{R}^n$  is a  $k$ -tensor field  $w$  s.t.  $w(x) \in A^k(T_x \mathbb{R}^n)$   $w(x) = \sum_{I \in \binom{n}{k}} a_I(x) \psi_I$

\*  $w$  is  $C^r$  if 1. ( $w = \sum a_I \phi_I$  or  $\sum a_I \psi_I$   $\forall I, a_I \in C^r$ )  $\Leftrightarrow$  2.  $(w, \gamma_1, \dots, \gamma_k)$  is  $C^r$  where  $\gamma_1, \dots, \gamma_k \in C^r$  v.f.)

\*  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ;  $w$ ,  $k$ -tensor field on  $\mathbb{R}^n$  then  $w(f \circ \gamma_1, \dots, f \circ \gamma_k) = f^* w(\gamma_1, \dots, \gamma_k) \in k$ -tensor field on  $\mathbb{R}^m$

$w$  is  $k$ -form  $\Rightarrow f^* w$  is  $k$ -form;  $f^* g^* w = (g \circ f)^* w$ ;  $f^*(\alpha w_1 + \beta w_2) = \alpha f^*(w_1) + \beta f^*(w_2)$

$\Omega^k(\mathbb{R}^n) / \Omega^k(M)$  = the vector space of all  $C^\infty$   $k$ -forms on  $\mathbb{R}^n / M$ .  $\Omega^k(A) =$  set of all  $k$ -forms on  $A$  ( $C^\infty$ )

Differential form of order 0:  $f : A \rightarrow \mathbb{R}$  of  $C^r$  where  $A$  is open in  $\mathbb{R}^n$  (also called a scalar field in  $A$ )

$(\omega \wedge \eta)(\gamma_1, \dots, \gamma_{k+l}) = (\omega(x) \wedge \eta(x))(v_1, \dots, v_{k+l})$  where  $\omega \in \Omega^k(M)$  &  $\eta \in \Omega^l(M)$

\* bilinear & associative; super-symmetric:  $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ ;  $f^*(\omega \wedge \eta) = (f^* \omega) \wedge (f^* \eta)$

$(dw)(\gamma_1, \dots, \gamma_{k+l}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{k+l}} w(\gamma(p(\gamma_1, \dots, \gamma_{k+l})))$  where  $w \in \Omega^k(\mathbb{R}^n)$ .

$$D_{\vec{z}} f = Df \cdot v$$



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\*  $f \in C^1(\mathbb{R}^n)$  then  $df = \sum_i \frac{\partial f}{\partial x_i} dx_i = \sum_i \frac{\partial f}{\partial x_i} \phi_i$

\*  $\exists$  linear operator  $d: \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n)$  s.t. if  $f \in \Lambda^0(\mathbb{R}^n)$  then  $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ ;  $3. d(dw) = 0$ .  
2.  $w \in \Lambda^k, \eta \in \Lambda^l, d(w \wedge \eta) = (dw) \wedge \eta + (-1)^k w \wedge (d\eta)$

$\phi: \mathbb{R}^k \rightarrow \mathbb{R}^m$   $\phi^*: \Lambda^k(\mathbb{R}^m) \rightarrow \Lambda^k(\mathbb{R}^n)$  by  $\phi^*(w_1, \dots, w_k) = w(\phi^* \xi_1, \dots, \phi^* \xi_k)$ , where  $\xi_i \in T_{\phi(x)} \mathbb{R}^m$

property: 1.  $\phi^*$  linear; 2.  $\phi^*(w \wedge \eta) = \phi^*(w) \wedge \phi^*(\eta)$ ; 3.  $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ ; 4.  $\phi^*(dw) = d(\phi^* w)$

\*  $\phi^*(w) = \phi^*(f dy_1) = \det(D\phi) \phi^* f dx_1$  where  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , &  $w = f dy_1 \in \Lambda^{top}(\mathbb{R}^m)$ .

\*  $w = \sum_{i \in I_1} a_i dy_1 \in \Lambda^k(\mathbb{R}^m)$  then  $\phi^* w = \sum_{j \in J_1} \sum_{i \in I_2} (\phi^* a_j \det(D\phi)_{j,i}) dx_j$ .  $(D\phi)_{j,i}$  is <sup>kk matrix by</sup> rows  $j$  & cols  $i$ .

Integral  $S_Q w := S_{\mathbb{R}^k} w := S_A f = S_Q f$  where  $w = f dx_1 \in \Lambda^{top}(\mathbb{R}^k)$  and  $\text{supp}(w) = \{x \in \mathbb{R}^k : w(x) \neq 0\} \subset Q$  rectangle.

Integral  $S_{Y_\alpha} w := S_A \alpha^* w = S_A f$  where  $\alpha^* w = f dx_1$ ;  $Y_\alpha$  = parametrized manifold where  $\alpha: A \rightarrow \mathbb{R}^n$  is  $C^\infty$  &  $Y = \alpha(A)$

\*  $S_{Y_\alpha} w = \int_{Y_\alpha} w$  where  $g: A \xrightarrow{\text{open}} B$  does not change sign on  $A$   $w \in \Lambda^{top}(M)$  where  $Y \subseteq M \subseteq \mathbb{R}^k$

Sign of  $\det(Dg)$  and  $\beta: B \rightarrow \mathbb{R}^m$   $C^\infty$  and  $\alpha = \beta \circ g$  where  $\alpha(A) = \beta(B) = Y$  and  $w \in \Lambda^k(M)$  where  $Y \subseteq M \subseteq \mathbb{R}^n$

\*  $S_{Y_\alpha} w = S_A (f \circ \alpha) \det\left(\frac{\partial \alpha^i}{\partial x^j}\right)$  if  $w = f dx_1 \in \Lambda^k(M)$  for open  $M$ .  $Y = \alpha(A) \subseteq M \subseteq \mathbb{R}^n$  where  $\alpha: A \rightarrow \mathbb{R}^n$  is  $C^\infty$  for open  $A \subseteq \mathbb{R}^k$

Orientation (on finite dim vector space) = choice of  $n$ -basis for  $V$  regard up to positive-det changes.

i.e.  $(v_1, \dots, v_n) \sim (u_1, \dots, u_n)$  if  $\det C_v^u > 0$  where  $C_v^u = (c_{ij})$   $u_i = \sum c_{ij} v_j$

Oriented vector space = f.d.v.s. along with a choice of orientation on it.

\* Every f.d.v.s  $V$  has exactly 2 orientation (distinct)

Orientation on manifold  $M$  is a continue choice of an orientation  $\phi_x$  on  $T_x M$  for all  $x \in M$ .

M is orientable if every  $p \in M$  has a nbd  $W$  with cont. vector fields  $u_1, \dots, u_k$  defined on  $W$  s.t.  $(u_1|_W, \dots, u_k|_W)$  is  $\Lambda^k$

\*  $M^k \subset \mathbb{R}^{k+1}$  and is oriented, then  $\exists$  bijection  $\{\text{orientation of } M\} \leftrightarrow \{\text{cont. varying choice of unit normal vectors to } M \text{ in } \mathbb{R}^{k+1}\}$

So  $\forall p \in M, n(p) \in T_p \mathbb{R}^{k+1}$  s.t. 1)  $n(p) \perp T_p M$  2)  $\|n(p)\| = 1$  3)  $p \mapsto n(p)$  is continuous.

\*  $M$  is oriented  $\Leftrightarrow \partial M$  is oriented.

\*  $\phi: M^k \rightarrow N^l$  is  $C^r$  with max rank differential then "orientation preserving" or "positive"  
 $\Leftrightarrow D\phi: T_p M \rightarrow T_{\phi(p)} N$  is orientation preserving.

\*  $M^k$  oriented,  $\alpha \& \beta$  positive &  $\text{supp } w \subset \text{im } \alpha \cap \text{im } \beta$  then  $S_{T_p M} \alpha^* w = S_M^\alpha w = S_M^\beta w = S_{T_p N} \beta^* w = S_N w$

$S_M w = \sum_{i=1}^k S_{M_i} \phi_i^* w$  where  $\{\phi_i\}$  is partition of unity on  $M$  dominated by coordinate patches covering  $M$  &  $\in$  orientation

\*  $\phi_i \& \psi_i$  are partition of unity then  $S_M^\phi w = S_M^\psi w$ .

\* For  $a_1 \in \mathbb{R}$ ,  $S_M a_1 w_1 + a_2 w_2 = a_1 S_M w_1 + a_2 S_M w_2$ ;  $S_M w = -S_{-M} w = S_{-M} - w$ .

\* (Stokes thm)  $M^k$  compact & oriented &  $w \in \Lambda^{k-1}(M)$  then  $S_M dw = S_{\partial M} w$ .

\* Every exact form is closed. On  $\mathbb{R}^n$  every closed form is exact. exact:  $\exists \lambda$ .  $w = d\lambda$

\*  $M$  compact. closed  $\Rightarrow H_{DR}^k(M) = \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-forms}\}$  which is finite dim.



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