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"$\Rightarrow$" Suppose L, R exist,

Suppose $\beta = \{u_1, \cdots u_n\}$ is a basis of V

Claim $(w_1 \doteq R(u_1), \cdots, w_n \doteq R(u_n))$ is a basis of W, hence

$\dim V = \dim W$.

Pf Given $w \in W$

set $u = L(w)$ as β is a basis

$\exists \alpha_1 \cdots \alpha_n$ s.t. $u = \sum \alpha_i u_i$

Apply R to both sides

$w = R(u) = R(\sum \alpha_i u_i) = \sum \alpha_i w_i$

So (w_i) span

Now suppose $\sum \alpha_i w_i = 0$

apply L to both sides

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$\sum \alpha_i L(w_i) = L(0)$

$\quad\shortparallel$

$\sum \alpha_i u_i = 0$

$\Rightarrow$ as u_i are lin indep. $\forall i\ \alpha_i = 0$ a

Cor If $\dim V = n$, then V is isomorph to F^n

choose a basis $\beta = \{u_1, \ldots u_n\}$ for V

use the standard basis $(e_1, \ldots e_n)$ for F

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} i$$

$$V \ni x \xrightarrow[R(x)]{\text{isomorphic}} [x]_\beta \in F^n$$

"the coordinates of x rel. β"

$$x = \sum \alpha_i u_i \Longleftrightarrow [x]_\beta = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$R(x) = \sum \alpha_i e_i$

~~Example~~ In $P_3(\mathbb{R})$

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$[(x-1)^3](1, x, x^2, x^3) =$
$\quad\quad\quad\quad u_1\, u_2\, u_3\;\; u_4$

$(x-1)^3 = x^3 - 3x^2 + 3x - 1$

$= u_4 - 3u_3 + 3u_2 - u_1$

Fix a l.t. $T: V \to W$.

Def. 1. $N(T) = \ker_V(T)$ | "null space" or "kernel" of T.
$= \{v \in V : T(v) = 0\}$

2. "Range" "image"

$R(T) = \mathrm{im}(T) = \{T(v) : v \in V\}$

Proposition / definition

1. $N(T) \subset V$ is a subspace of V
$\dim N(T) = \mathrm{nullity}(T)$

2. $R(T) \subset W$ is a subspace of W
$\dim R(T) = \mathrm{rank}(T)$