R3: existence of units

additive unit $\rightarrow$ $a+0=a$.

multiplicative unit $\rightarrow$ $a \cdot 1 = a$.

R4 "The existence of inverses"

$\forall a \in \mathbb{R}$ $\exists b \in \mathbb{R}$ s.t. $a+b=0$.

"there exists"

$\forall a \in \mathbb{R}$ $a \neq 0 \Rightarrow \exists b$ s.t. $a \cdot b = 1$

($\forall a \neq 0$ $\exists b$ s.t. ....).

R5. "The distributive law"

$\forall a, b, c$ $(a+b)c = ac + bc$

$(a+b)(a-b) = a^2 - b^2$

follows from R1 – R5

True for $\mathbb{R}$, yet does not follow from R1 to R5.

$\forall a$ $\exists x$ s.t. $a = x^2$ or $(-a = x^2)$ $x^2 + a = 0$.

"inclusive or"

in $\mathbb{Q}$ take $a = 2$ $\nexists x$ s.t. $x^2 = 2$ or $x^2 + 2 = 0$.

Definition A "Field" is a set F along with a pair of binary operations

$+, \times : F \times F \to F$ and along with a pair

$0, 1 \in F$, $0 \neq 1$

& s.t. F1–F5 hold.

Examples 1. $\mathbb{R}$ is a field

2. $\mathbb{Q}$ = rational numbers is a Field

3. $\mathbb{C}$ "complex numbers" is a Field

4. $F = \{\overset{\text{even}}{0}, \overset{\text{odd}}{1}\} = F_2 = \mathbb{Z}/2$.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Proposition F is a field.