

Friday Jan 27<sup>th</sup>, Hour 046

Manifolds and Boundaries, Integration on Boundaries

Read along: 24 - 25

$$\mathbb{R}^2_{xy} \rightarrow \mathbb{R}^3$$

$$x^2 - y^2 = 0 \Leftrightarrow (x+y)(x-y) = 0$$

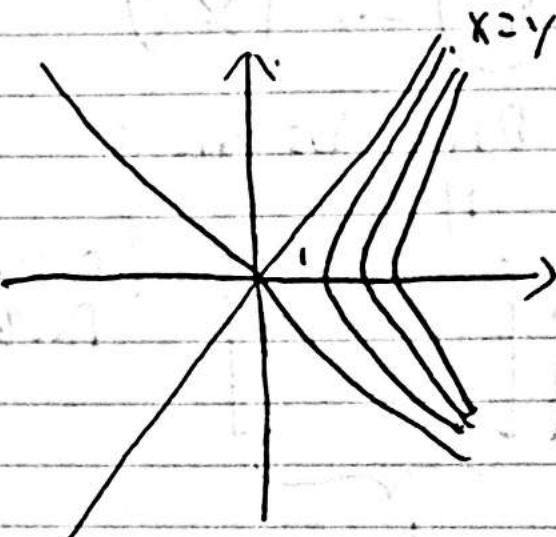
$$x \mapsto \begin{pmatrix} x \\ y \\ x^2 - y^2 \end{pmatrix}$$

Diagram: x-axis, y-axis.  $F(x,y) = x^2 - y^2$

$$x^2 - y^2 = 1 \Rightarrow y = \sqrt{x^2 - 1}$$

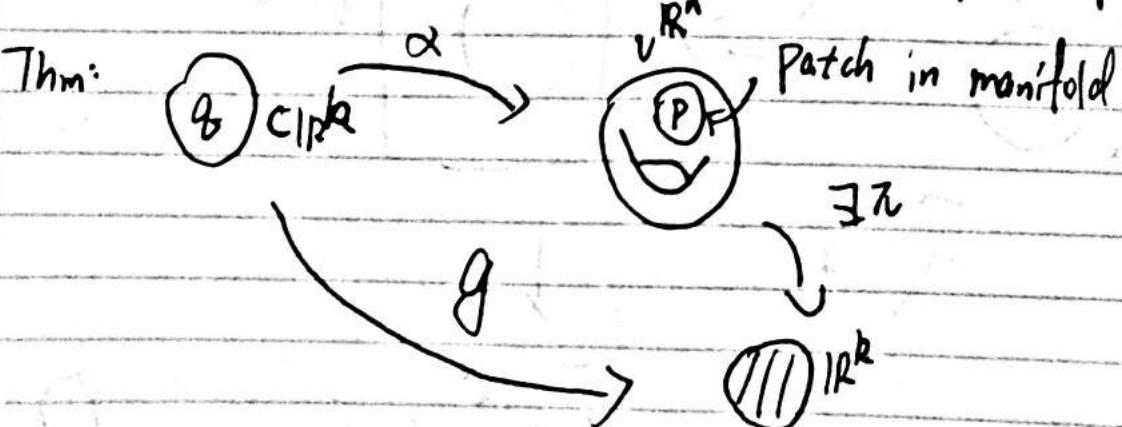
so  $x$  large, converges to  $y$ .

finally becomes



Or. Ex.  $(\alpha, \beta) \mapsto \left( \frac{\cos \alpha}{\sin \beta}, \frac{(2 + \cos \beta)}{\sin \beta} \right) =$

Will be a good  $\mathbb{R}^2$  manifolds if pick a right angle.



Given a patch  $\alpha: V \rightarrow M \subset \mathbb{R}^n$ , and any  $p = \alpha(a)$ , there exists a projection  $\pi_0: \mathbb{R}^n \rightarrow \mathbb{R}^k$  (defined by forgetting  $(n-k)$  of the coordinates of  $\mathbb{R}^n$ ) such that  $g = \pi_0 \circ \alpha$ , then,  $g$  is a diffeomorphism in the infinity of  $a$ .

Pf: By def of a "patch"  $d\alpha(g)$  has rank  $K$ .

$\left\{ \underbrace{\begin{bmatrix} d\alpha(g) \\ \vdots \\ d\alpha(g) \end{bmatrix}}_k \right\}$  So it has  $k$  linearly independent rows.  
For convenience, assume there are the first  $k$  rows  
and let  $\Pi: \mathbb{R}^n \rightarrow \mathbb{R}^k$  be given by dropping the last  
( $n-k$ ) coordinates of a vector in  $\mathbb{R}^n$ , then

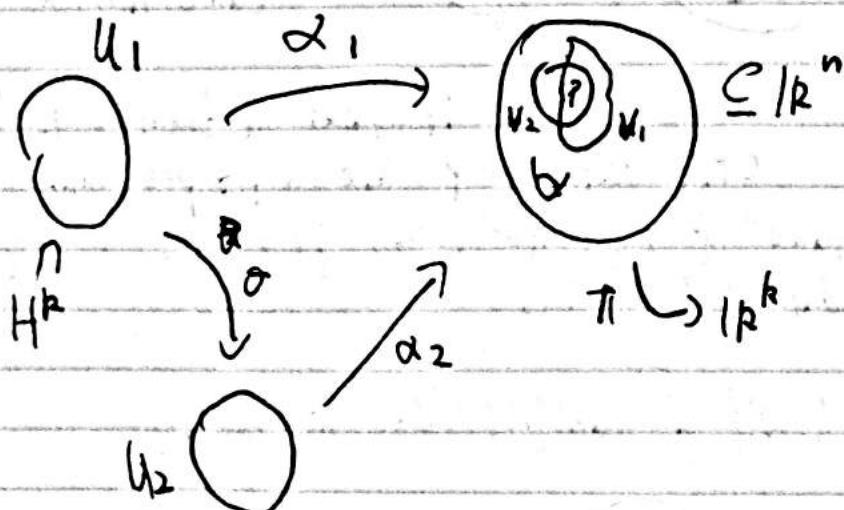
$$d\Pi = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_k$$

Set  $g = \pi \circ \alpha$ ,  $dg = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_k \underbrace{(d\alpha)}_{n-k} =$  first  $k$  or  $k$  rows of  $d\alpha$

So  $dg(g)$  has  $k$  independent rows. so  $dg(\alpha)$  is invertible

by the IFT,  $g$  is a diffamorphism in the vicinity  $A \rightarrow g(A)$ .

Corollary: Transition functions are  $C^r$



Claim:  $\sigma$  is of  $C^r$

$\sigma = \alpha_2^{-1} \circ \alpha_1$  on the set of  $\pi^{-1}(U_1 \cap U_2)$  ✓ Why true?

Pf: find a projection  $\Pi: \mathbb{R}^n \rightarrow \mathbb{R}^k$  such that  $g = \pi \circ \alpha_2$  is diffamorphism near some arbitrary point  $p \in U_1 \cap U_2$

Claim:  $\sigma = \alpha_2^{-1} \circ \alpha_1 = g^{-1} \circ \pi \circ \alpha_1$  (near  $g_p$ )

Pf: proven by composing with  $f$

$$d_2^{-1} \circ d_1 = f^{-1} \circ \pi \circ \alpha, \text{ ints}$$

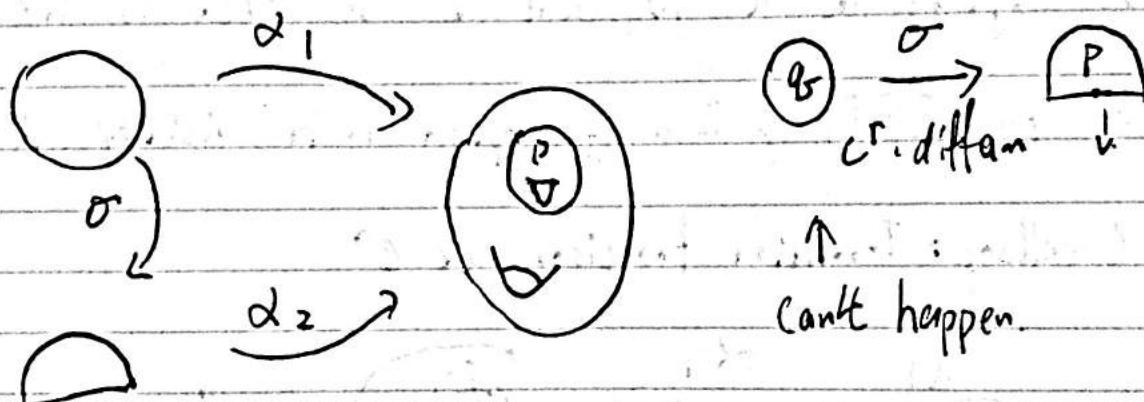
$$f \circ d_2^{-1} \circ d_1 = f \circ f^{-1} \circ \pi \circ \alpha,$$

$$f \circ d_2^{-1} \circ \alpha = \pi \circ \alpha, \text{ recall } f = \pi \circ \alpha$$

$$\pi \circ d_2 \circ d_2^{-1} \circ \alpha = \pi \circ \alpha, \text{ so } \sigma = f^{-1} \circ \pi \circ \alpha \text{ is a}$$

composition of  $C^r$  functions. hence it is  $C^r$  as required

Corollary:  $\partial M$  makes sense.



it requires differential  $q$ , invertible

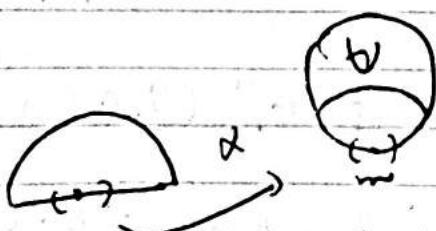
(take vectors in  $P$ , have a pre image in  $Q$ )

take small distance of the vector in  $q$ , go out of the space in  $p$ .

Then contradiction arises

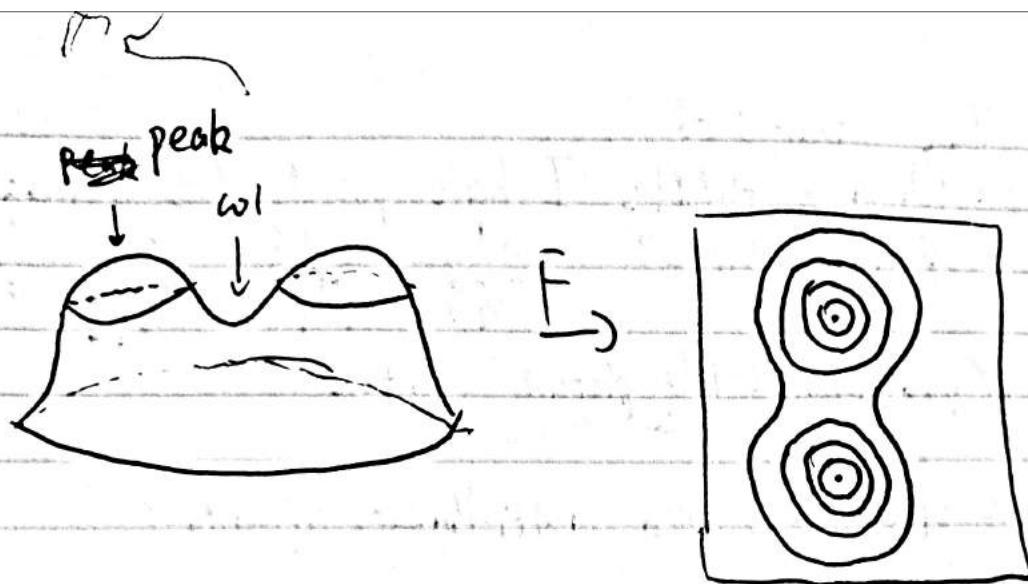
Early  $\partial M$  is itself a manifold, with no boundary

Sketch:



$d$ : restricted edge, is a patch of  $\partial M$

Every ~~not~~ point at boundary has a nbd that has a coordinate patch, defined in this way



If  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^r$  and  $t$  is a height at which  $dF$  has no singularities, then  $F^{-1}(t)$  is a manifold (circles)