

Jan 13th Read along Sec. 21

Gram-Schmidt

$$V_1' = u_1 \quad V_1 = \frac{V_1'}{\|V_1'\|}$$
$$V_2' = u_2 - \langle u_2, V_1 \rangle V_1 \quad V_2 = \frac{V_2'}{\|V_2'\|}$$
$$\vdots$$
$$V_k' = u_k - \sum_{i=1}^{k-1} \langle u_k, V_i \rangle V_i \quad V_k = \frac{V_k'}{\|V_k'\|}$$

Theorem: There's a unique $V: (\mathbb{R}^n)^k \rightarrow \mathbb{R}_+$ s.t.

1. If $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an orthogonal transformation, & $x_i \in \mathbb{R}^n$
 $V(h(x_1), \dots, h(x_k)) = V(x_1, \dots, x_k)$
2. If $x \in \mathbb{R}^k \times \{0\}$, so $x_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix}$, $y_i \in \mathbb{R}^k$, then
 $V(x_1, \dots, x_k) = |\det(y_1, \dots, y_k)|$

Furthermore,

3. $V(x_1, \dots, x_k) = 0 \Leftrightarrow \{x_i\}$ is lin dependent
4. if $X = (x_1 | \dots | x_k) \in M_{n \times k}$
then $V(x_1, \dots, x_k) = |\det(X^T X)|^{1/2}$

Pf We want to show that 1-2 above determine $V(x_1, \dots, x_k)$

Let $W = \text{span}(x_1, \dots, x_k)$, find an orthonormal basis $(f_i)_{i=1}^l$ of W
where $l = \dim(W)$, $l \leq k$.

Extend $(f_i)_{i=1}^l$ to an orthonormal basis $(f_i)_{i=1}^n$ of \mathbb{R}^n .

Let $A = (f_1 | f_2 | \dots | f_n)$ so, $Ae_i = f_i$.

A is orthogonal, so it's invertible; with an orthonormal inverse.

Let h be the linear transformation represented by A^{-1}

It is orthogonal, and $h(f_i) = e_i$ so $h(f_1), \dots, h(f_l) \in \mathbb{R}^l \subset \mathbb{R}^k$

Now $V(x_1, \dots, x_k) = V(h(x_1), \dots, h(x_k))$,

but each $x_i \in W$, $h(f_i)_{i=1}^l \in \mathbb{R}^k \Rightarrow h(W) \in \mathbb{R}^k$

so $h(x_i) \in \mathbb{R}^k$, and the R.H.S is determined by (2).

For existence, need to show $V(x_1, \dots, x_k) = |\det(X^T X)|^{1/2}$ implies 1 and 2

1. Suppose h is orthogonal, meaning $h(x) = Ax$, where $A^T A = I$

$$V(h(x_1), \dots, h(x_k)) = V(Ax_1, \dots, Ax_k)$$

$$X_h = (Ax_1 | \dots | Ax_k) = |\det(X_h^T X_h)|^{1/2} = |\det(X^T A^T A X)|^{1/2}$$

$$= A \cdot (x_1, \dots, x_k) = A \cdot |\det(X^T X)|^{1/2} = V(x_1, \dots, x_k)$$

2. Suppose $X_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix}_{n-k}^k$

$$X = (X_1 | \dots | X_k) = \begin{pmatrix} y_1 & \dots & y_k \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} y_1 & \dots & y_k \\ 0 & \dots & 0 \end{pmatrix}_{n-k}^k$$
$$= \begin{pmatrix} Y \\ 0 \end{pmatrix} = |\det(Y)|^{1/2} = |\det(Y)|$$

3. $\{X_i\}$ dependent

$$\exists a \neq 0 \text{ s.t. } X \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} = 0 \Rightarrow X^T X a = 0, X^T X \text{ is not invertible}$$

$$\text{so } V = |\det(X^T X)|^{1/2} = 0$$

$$\Rightarrow V(x_1, \dots, x_k) = 0$$

$$\det(X^T X) = 0, \text{ so } \exists a \neq 0 \text{ s.t. } X^T X a = 0$$

$$a^T X^T X a = 0$$

$$(Xa)^T \cdot Xa = 0 \Rightarrow \|Xa\|^2 = 0$$

So $Xa = 0$, so cols of X are l.d.

Example 2 \subset 3 $x, y \in \mathbb{R}^3$

$$V(x, y) = |\det XX^T|^{1/2} = \left| \det \begin{pmatrix} \|x\|^2 & \langle x, y \rangle \\ \langle y, x \rangle & \|y\|^2 \end{pmatrix} \right|^{1/2}$$
$$= \left| \|x\|^2 \|y\|^2 - \langle x, y \rangle^2 \right|^{1/2}, \langle x, y \rangle^2 = \|x\|^2 \|y\|^2 \cos^2 \theta$$
$$= \|x\|^2 \|y\|^2 (1 - \cos^2 \theta)^{1/2}$$
$$= \|x\|^2 \|y\|^2 \sin^2 \theta^{1/2}$$
$$= \|x\| \|y\| |\sin \theta| \rightarrow \text{Area of parallelogram}$$