5. (h) Solution: No

Now we need to show it.

Suppose that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is in the span of S

We know that $\text{span}(S) = \left\{ M : M = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \ \forall a,b,c \in \mathbb{R} \right\}$.

Then from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{span}(S)$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad a,b,c \in \mathbb{R}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a+c & b+c \\ -a & b \end{pmatrix}$$

Then we get a system of linear equations:

$$\begin{cases} a + c = 1 & ① \\ b + c = 1 & ② \\ -a = 0 & ③ \\ b = 1 & ④ \end{cases}$$

① + ③, ② − ④:

$$\Rightarrow \begin{cases} c = 1 & ① \\ c = 0 & ② \\ a = 0 & ③ \\ b = 1 & ④ \end{cases} \Rightarrow \begin{cases} c = 1 & ① \\ 0 = -1 & ② \\ a = 0 & ③ \\ b = 1 & ④ \end{cases}$$

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Then there is a contradiction in ②

$\Rightarrow$ There is no solution to this system of linear equations.

Hence $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the span(S).

10. Proof:

We know that:

$\forall M \in span\{M_1, M_2, M_3\}$, there exists a pair of (a, b, c) $(a, b, c \in \mathbb{R})$, s.t.

$M = aM_1 + bM_2 + cM_3$

$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$

$= \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$\because a_{12} = c \quad a_{21} = c$

$\therefore a_{12} = a_{21} = c$

$\therefore \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is symmetric 2x2 matrice.

$\Rightarrow$ All the vectors in $span\{M_1, M_2, M_3\}$ are symmetric 2x2 metrices

Now we need to prove that all symmetric 2x2 matrices are in $span\{M_1, M_2, M_3\}$

Suppose that there is a symmetric 2x2 matrix M' which is not in the span$\{M_1, M_2, M_3\}$.

$M' = \begin{pmatrix} a' & c' \\ c' & b' \end{pmatrix}$.

But we find that:

$a'\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b'\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix} + \begin{pmatrix} 0 & c' \\ c' & 0 \end{pmatrix}$

$= \begin{pmatrix} a' & c' \\ c' & b' \end{pmatrix} = M'$

which means that M' can be generated by M_1, M_2, M_3, hence it is in span $\{M_1, M_2, M_3\}$

There is a contradiction.

Hence the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2x2 matrices.

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